高等数学习题课

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1. 曲线的切线

2.曲面的法向量

3.梯度

4.方向导数

5.二元函数的泰勒展开

6.最小二乘法

空间曲面的切平面与法线方程

空间曲面方程为： $F(x,y,z)=0$

曲面Σ在点 $M_0(x_0, y_0, z_0)$处的切平面方程为

$$F_x'(x_0, y_0, z_0)(x-x_0)+F_y'(x_0, y_0, z_0)(y-y_0)+F_z'(x_0, y_0, z_0)(z-z_0)=0$$

曲面Σ 在点 M_0 处的法线方程为

$$\frac{x-x_0}{F_x'(x_0,y_0,z_0)}=\frac{y-y_0}{F_y'(x_0,y_0,z_0)}=\frac{z-z_0}{F_z'(x_0,y_0,z_0)}.$$

梯度：

$$c=\nabla f(\mathbf{x}^*)=\begin{bmatrix}\dfrac{\partial f(\mathbf{x}^*)}{\partial x_1}\\ \dfrac{\partial f(\mathbf{x}^*)}{\partial x_2}\\ \vdots\\ \dfrac{\partial f(\mathbf{x}^*)}{\partial x_n}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial f(\mathbf{x}^*)}{\partial x_1} & \dfrac{\partial f(\mathbf{x}^*)}{\partial x_2} & \cdots & \dfrac{\partial f(\mathbf{x}^*)}{\partial x_n}\end{bmatrix}^T$$

Hessian矩阵：

$$\frac{\partial^2 f}{\partial \mathbf{x} \partial \mathbf{x}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

若$f(X)$具有二阶连续偏导，则$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$，

则Hessian矩阵为对称矩阵。

方向导数的计算

若函数$u=f(x,y,z)$在点P_0 $(x_0$、y_0、$z_0)$处可微，

则函数f在点P_0处沿任意方向的方向导数为，

$$\frac{\partial f}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma$$

即 $$\frac{\partial f}{\partial l}=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\bullet(\cos\alpha,\cos\beta,\cos\gamma)$$

L方向的单位向量

一元函数$y=f(x)$的泰勒(Taylor)公式，

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$+\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}\quad(0<\theta<1)$$

可用n次多项式来近似表达函数$f(x)$，且误差是

当$x\to x_0$时比 $(x-x_0)^n$高阶的无穷小.

下面将考虑用多个变量的多项式来近似表达一个

给定的多元函数，并能具体地估算出误差的大小.

二元函数的泰勒(Taylor)公式，

定理 若函数$z=f(x,y)$在点$P_0(x_0,y_0)$的某一邻域$U(P)$内具有直至$n+1$阶的连续偏导数，点$(x_0+h,y_0+k)\in U(P)$则有

$$f(x_0+h,y_0+k)=f(x_0,y_0)+(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f(x_0,y_0)$$

$$+\frac{1}{2!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2 f(x_0,y_0)+\cdots+\frac{1}{n!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^n f(x_0,y_0)+R_n \quad (1)$$

$$R_n=\frac{1}{(n+1)!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1} f(x_0+\theta h,y_0+\theta k) \quad (0<\theta<1)$$

其中记号

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)=hf_x(x_0,y_0)+kf_y(x_0,y_0)$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)=h^2 f_{xx}(x_0,y_0)+2hkf_{xy}(x_0,y_0)+k^2 f_{yy}(x_0,y_0)$$

一般地，记号 $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f(x_0,y_0)$ 表示为

$$\sum_{p=0}^{m} C_m^p h^p k^{m-p} \frac{\partial^m f}{\partial x^p \partial y^{m-p}}\bigg|_{(x_0,y_0)}$$

利用一元函数的泰勒公式进行证明.

一元函数泰勒展开：

$$f(x)=f(x^*)+\frac{df(x^*)}{dx}(x-x^*)+\frac{1}{2}\frac{d^2f(x^*)}{dx^2}(x-x^*)^2+R$$

$x-x^*=d$ (a small change in the point x^*).

$$f(x^*+d)=f(x^*)+\frac{df(x^*)}{dx}d+\frac{1}{2}\frac{d^2f(x^*)}{dx^2}d^2+R$$

二元函数泰勒展开：

$$f(x_1, x_2) = f(x_1^*, x_2^*) + \frac{\partial f}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x_2 - x_2^*)$$
$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*)^2\right] + R$$

$$f(x_1, x_2) = f(x_1^*, x_2^*) + \sum_{i=1}^{2}\frac{\partial f}{\partial x_i}(x_i - x_i^*)$$
$$+ \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_i - x_i^*)(x_j - x_j^*) + R$$

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f^T(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x} - \mathbf{x}^*) + R$$

n 元函数泰勒展开：

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f^T(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x} - \mathbf{x}^*) + R$$

$$f(\mathbf{x}^* + \mathbf{d}) = f(\mathbf{x}^*) + \nabla f^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}\mathbf{d} + R$$

求曲面$x^2+y^2+z^2=2$ 上平行于平面 $x+4y-z=0$ 的切平面方程。

求曲面 $z=x^2+y^2$在点(1, 1, 2)处的切平面方程。

在点(1, 2)处，计算多元函数的梯度和Hessian矩阵：

$$f(\mathbf{x})=x_1^3+x_2^3+2x_1^2+3x_2^2-x_1x_2+2x_1+4x_2$$

求函数$u=ln(x+y^2+z^2)$在点M_0(0，1，2)处，沿向量(2,-1,-1)的方向的方向导数。

求曲面$x^2+y^2+z^2=2$ 上平行于平面 $x+4y-z=0$ 的切平面方程。

解： 空间曲面法向量

设函数$F(x,y,z)=x^2+y^2+z^2-2$,

则 $n=(F_x,F_y,F_z)=(2x,2y,2z)$

设切点为(x_0, y_0, z_0), 则

$$\frac{x_0}{1}=\frac{y_0}{4}=\frac{z_0}{-1}$$

$$x_0^2+y_0^2+z_0^2=2$$

切点为(1/3,4/3,-1/3),(-1/3,-4/3,1/3)

切面方程为：

$$(x-\frac{1}{3})+4(y-\frac{4}{3})-(z+\frac{1}{3})=0 \text{ 或 } (x+\frac{1}{3})+4(y+\frac{4}{3})-(z-\frac{1}{3})=0$$

即 $x+4y-z=6$ 或 $x+4y-z=-6$

求曲面 $z = x^2 + y^2$在点(1, 1, 2)处的切平面方程。

解： 设函数$F(x, y, z) = x^2 + y^2 - z$,

则 $n = (F_x, F_y, F_z) = (2x, 2y, -1)$

切点为(1, 1, 2), 则

n=(2,2,-1)

$$2(x-1) + 2(y-1) - 1(z-2) = 0$$

$$2x + 2y - z - 2 = 0$$

练习：

在点(1, 2)处，计算多元函数的梯度和Hessian矩阵：

$$f(\mathbf{x}) = x_1^3 + x_2^3 + 2x_1^2 + 3x_2^2 - x_1x_2 + 2x_1 + 4x_2$$

解：

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 - x_2 + 2; \quad \frac{\partial f}{\partial x_2} = 3x_2^2 + 6x_2 - x_1 + 4$$

$\mathbf{c} = (7, 27).$

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4; \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -1; \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = -1; \quad \frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 6.$$

$$\mathbf{H}(1,2) = \begin{bmatrix} 10 & -1 \\ -1 & 18 \end{bmatrix}$$

练习：

求函数$u=ln(x+y^2+z^2)$在点M_0(0，1，2)处，沿向量(2,-1,-1)的方向的方向导数。

解：

所给向量的方向余弦为$\left(\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$

点M_0处的梯度为$\left(\frac{1}{5},\frac{2}{5},\frac{4}{5}\right)$

则M_0处，沿所给向量的方向导数为$-\frac{4}{5\sqrt{6}}$

1. 求函数 $f(x,y)=2x^2-xy-y^2-6x-3y+5$ 在点 $(1,-2)$ 的泰勒公式.

解: $f(1,-2)=5$, $f'_x(1,-2)=(4x-y-6)\Big|_{(1,-2)}=0$, $f'_y(1,-2)=(-x-2y-3)\Big|_{(1,-2)}=0$,

$f''xx(1,-2)=4$, $f''_{xy}=-1$, $f''_{yy}(1,-2)=-2$.

又阶数为 3 的各偏导数为零,则

$$
\begin{aligned}
f(x,y)&=f[1+(x-1),-2+(y+2)]\\
&=f(1,-2)+(x-1)f'_x(1,-2)+(y+2)f'_y(1,-2)+\\
&\quad\frac{1}{2!}[(x-1)^2f''_{xx}(1,-2)+2(x-1)(y+2)f''_{xy}(1,-2)+(y+2)^2f''_{yy}(1,-2)]\\
&=5+\frac{1}{2!}[4(x-1)^2-2(x-1)(y+2)-2(y+2)^2]\\
&=5+2(x-1)^2-(x-1)(y+2)-(y+2)^2.
\end{aligned}
$$

1. 某种合金的含铅量百分比(%)为 p,其溶解温度(℃)为 θ,由实验测得 p 与 θ 的数据如下表:

$p/\%$	36.9	46.7	63.7	77.8	84.0	87.5
$\theta/℃$	181	197	235	270	283	292

试用最小二乘法建立 θ 与 p 之间的经验公式 $\theta=ap+b$.

解: 由方程组 $\begin{cases} a\sum\limits_{i=1}^{6}p_i^2+b\sum\limits_{i=1}^{6}p_i=\sum\limits_{i=1}^{6}\theta_i p_i, \\ a\sum\limits_{i=1}^{6}p_i+6b=\sum\limits_{i=1}^{6}\theta_i, \end{cases}$ 确定先验公式中的 a,b,先算出

$$\begin{cases} \sum\limits_{i=1}^{6}p_i^2=28\ 365.28, & \sum\limits_{i=1}^{6}p_i=396.6, \\ \sum\limits_{i=1}^{6}\theta_i p_i=101\ 176.3, & \sum\limits_{i=1}^{6}\theta_i=1\ 458, \end{cases}$$

代入方程组,得 $\begin{cases} 28\ 365.28a+396.6b=101\ 176.3, \\ 396.6a+6b=1\ 458. \end{cases}$ 解此方程组,得 $a=2.234$,$b=95.33$.

则经验公式 $\theta=2.234p+95.33$.

2. 已知一组实验数据为(x_1,y_1)，(x_2,y_2)，$\cdots$，(x_n,y_n). 现若假定经验公式是 $y=ax^2+bx+c$. 试按最小二乘法建立 a,b,c 应满足的三元一次方程组.

解：设 M 是每个数据差的平方和：$M=\sum\limits_{i=1}^{n}[y_i-(ax_i^2+bx_i+C)]^2=M(a,b,c)$，令

$$\begin{cases}\dfrac{\partial M}{\partial a}=-2\sum\limits_{i=1}^{n}[y_i-(ax_i^2+bx_i+C)](x_i)^2=0,\\ \dfrac{\partial M}{\partial b}=-2\sum\limits_{i=1}^{n}[y_i-(ax_i^2+bx_i+C)](x_i)=0,\\ \dfrac{\partial M}{\partial c}=-2\sum\limits_{i=1}^{n}[y_i-(ax_i^2+bx_i+C)]=0,\end{cases}$$

即 $\begin{cases}\sum\limits_{i=1}^{n}(y_ix_i^2-ax_i^4-bx_i^3-cx_i^2)=0,\\ \sum\limits_{i=1}^{n}(y_ix_i-ax_i^3-bx_i^2-cx_i)=0,\\ \sum\limits_{i=1}^{n}(y_i-ax_i^2-bx_i-c)=0,\end{cases}$ 即 $\begin{cases}a\sum\limits_{i=1}^{n}x_i^4+b\sum\limits_{i=1}^{n}x_i^3+c\sum\limits_{i=1}^{n}x_i^2=\sum\limits_{i=1}^{n}x_i^2y_i,\\ a\sum\limits_{i=1}^{n}x_i^3+b\sum\limits_{i=1}^{n}x_i^2+c\sum\limits_{i=1}^{n}x_i=\sum\limits_{i=1}^{n}x_iy_i,\\ a\sum\limits_{i=1}^{n}x_i^2+b\sum\limits_{i=1}^{n}x_i+nc=\sum\limits_{i=1}^{n}y_i.\end{cases}$