高等数学习题课

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多元函数全微分的求法：

$$\mathrm{d}z = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

若 $u = f(x, y, z)$可微, 则

$$\mathrm{d}u = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

$$(4)u = x^{\frac{y}{z}}$$

解(4) $\because \dfrac{\partial u}{\partial x} = \dfrac{y}{z} x^{\frac{y}{z}-1}$

$$\frac{\partial u}{\partial y} = \ln x \cdot x^{\frac{y}{z}} \cdot \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = \ln x \cdot x^{\frac{y}{z}} \cdot \left(-\frac{y}{z^2}\right)$$

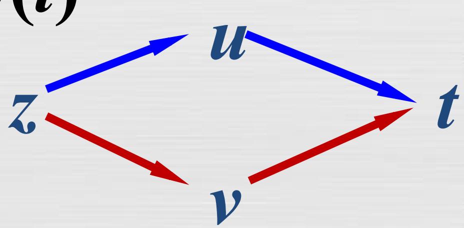
$$\therefore \mathrm{d}u = \frac{y}{z} x^{\frac{y}{z}-1} \mathrm{d}x + \ln x \cdot x^{\frac{y}{z}} \cdot \frac{1}{z} \mathrm{d}y + \ln x \cdot x^{\frac{y}{z}} \cdot \left(-\frac{y}{z^2}\right) \mathrm{d}z.$$

1、 复合函数的求导法则

(1) 复合函数只有一个自变量的情形的求导法则

如果函数$z=f(u,v),u=\varphi(t),v=\psi(t)$

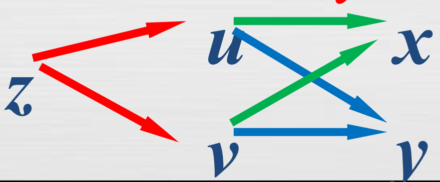
$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt},$$



(2)复合函数有两个自变量的情形的求导法则

如果函数$z=f(u,v),\ u=\varphi(x,y),\ v=\psi(x,y),$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\qquad\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$



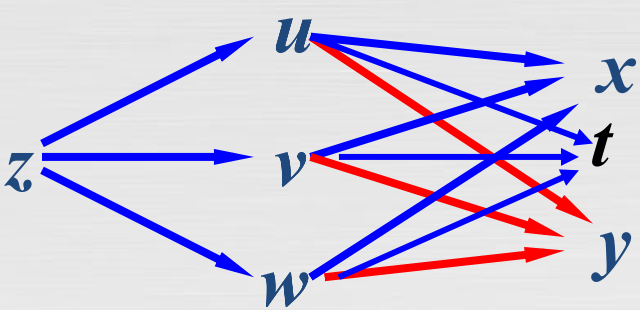
(3)复合函数有3个自变量的情形的求导法则

如果函数$z=f(u,v,w)$, $u=u(x,y,t)$, $v=v(x,y,t)$,

$w=w(x,y,t)$.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$

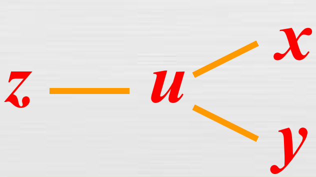
$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial t}$$



(4)复合函数有1个中间变量，两个自变量的求导法则

如果函数$z=f(u)$, $u=u(x,y)$

$$\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}$$

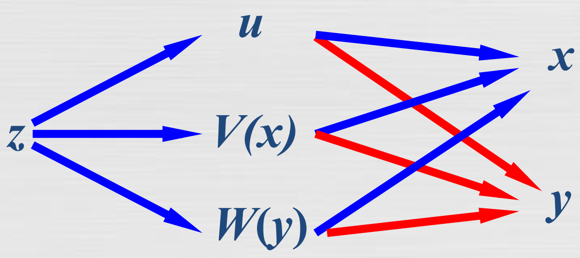


型

(5)复合函数的中间变量既有一元函数又有多元函数

如果函数$z = f(u, x, y)$,其中$u = \varphi(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}.$$



2 、多元复合函数的高阶导数的求导法则

注意：多元抽象复合函数求导时，常用导数符号.

求下列复合函数的偏导数或全导数:

(2) $z=\arctan\dfrac{x}{y}$, $x=u+v$, $y=u-v$, 求 $\dfrac{\partial z}{\partial u},\dfrac{\partial z}{\partial v}$

解(2)

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}+\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)$$

$$=\frac{y-x}{x^2+y^2}=\frac{-v}{u^2+v^2}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}+\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{-x}{y^2}\right)\cdot(-1)$$

$$=\frac{y+x}{x^2+y^2}=\frac{u}{u^2+v^2}.$$

$$(3)u=\ln(\mathrm{e}^x+\mathrm{e}^y),\qquad y=x^3$$，求 $\frac{\mathrm{d}u}{\mathrm{d}x}$

解(3) $$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\partial u}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}x}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$=\frac{1}{\mathrm{e}^x+\mathrm{e}^y}\cdot\mathrm{e}^x+\frac{1}{\mathrm{e}^x+\mathrm{e}^y}\cdot\mathrm{e}^y\cdot 3x^2$$

$$=\frac{\mathrm{e}^x+3x^2\mathrm{e}^y}{\mathrm{e}^x+\mathrm{e}^y}$$

$$=\frac{\mathrm{e}^x+3x^2\mathrm{e}^{x^3}}{\mathrm{e}^x+\mathrm{e}^{x^3}}$$

3.设 $z = xy + xF(u), u = \dfrac{y}{x}$ **$F(u)$为可导函数，证明:**

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy$$

证: $$\frac{\partial z}{\partial x} = y + xF'(u)\cdot\left(-\frac{y}{x^2}\right) + F(u) = F(u) + y - \frac{y}{x}F'(u)$$

$$\frac{\partial z}{\partial y} = x + xF'(u)\cdot\frac{1}{x} = x + F'(u)$$

故 $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left[F(u) + y - \frac{F'(u)y}{x}\right] + y[x + F'(u)]$$

$$= xF(u) + xy - yF'(u) + xy + yF'(u)$$

$$= xy + xF(u) + xy$$

$$= z + xy.$$

4.设 $z=\dfrac{y}{f(x^2-y^2)}$，其中$f(u)$为可导函数，验证

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$

证：$\because \dfrac{\partial z}{\partial x}=-\dfrac{yf'\cdot 2x}{f^2}=-\dfrac{2xyf'}{f^2}$

$$\frac{\partial z}{\partial y}=\frac{f-y\cdot f'\cdot(-2y)}{f^2}=\frac{f+2y^2f'}{f^2}$$

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=-\frac{2yf'}{f^2}+\frac{f+2y^2f'}{yf^2}=\frac{1}{yf}$$

$$=\frac{y}{f}\cdot\frac{1}{y^2}=\frac{z}{y^2}$$

5. $z=f(x^2+y^2)$，其中f具有二阶导数，求 $\dfrac{\partial^2 z}{\partial x^2},\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial y^2}$

解： $\dfrac{\partial z}{\partial x}=2xf',\quad \dfrac{\partial z}{\partial y}=2yf',$

$$\frac{\partial^2 z}{\partial x^2}=2f'+2x\cdot 2xf''=2f'+4x^2f'',$$

$$\frac{\partial^2 z}{\partial x\partial y}=2xf''\cdot 2y=4xyf'',$$

由对称性知，$\dfrac{\partial^2 z}{\partial y^2}=2f'+4y^2f''.$

6.设f是具有连续二阶偏导函数,求下列函数的二阶偏导数:

$$(1)\ z = f(x, \frac{x}{y})$$

解(1) $\frac{\partial z}{\partial x} = f_1' \cdot 1 + f_2' \cdot \frac{1}{y}$

$$= f_1' + \frac{1}{y} f_2',$$

$$\frac{\partial^2 z}{\partial x^2} = f_{11}'' + f_{12}'' \cdot \frac{1}{y} + \frac{1}{y}\left(f_{21}'' + f_{22}'' \cdot \frac{1}{y} \right)$$

$$= f_{11}'' + f_{12}'' \cdot \frac{2}{y} + \frac{1}{y^2} f_{22}'',$$

$$\frac{\partial^2 z}{\partial x \partial y} = f_{12}'' \cdot \left(-\frac{x}{y^2} \right) - \frac{1}{y^2} f_2' + \frac{1}{y} f_{22}'' \cdot \left(-\frac{x}{y^2} \right)$$

$$= -\frac{x}{y^2}\left(f_{12}'' + \frac{1}{y} f_{22}'' \right) - \frac{1}{y^2} f_2',$$

$$(1)\ z = f\left(x, \frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = f_2'\left(-\frac{x}{y^2}\right)$$

$$= -\frac{x}{y^2} f_2',$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3} f_2' - \frac{x}{y^2} f_{22}'' \cdot \left(-\frac{x}{y^2}\right)$$

$$= \frac{2x}{y^3} f_2' + \frac{x^2}{y^4} f_{22}''.$$

$$(3) z=f(\sin x,\cos y,\mathrm{e}^{x+y})$$

解(3) $\dfrac{\partial z}{\partial x}=f_1'\cdot\cos x+f_3'\cdot\mathrm{e}^{x+y}$

$$=\cos xf_1'+\mathrm{e}^{x+y}f_3',$$

$$\frac{\partial^2 z}{\partial x^2}=-\sin xf_1'+\cos x\left(f_{11}''\cdot\cos x+f_{13}''\cdot\mathrm{e}^{x+y}\right)$$

$$+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}\left(f_{31}''\cos x+f_{33}''\cdot\mathrm{e}^{x+y}\right)$$

$$=\mathrm{e}^{x+y}f_3'-\sin xf_1'+\cos^2 xf_{11}''+2\mathrm{e}^{x+y}\cos xf_{13}''+\mathrm{e}^{2(x+y)}f_{33}'',$$

$$\frac{\partial^2 z}{\partial x\partial y}=\cos x\left[f_{12}''\cdot(-\sin y)+f_{13}''\cdot\mathrm{e}^{x+y}\right]$$

$$+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}\cdot\left[f_{32}''\cdot(-\sin y)+f_{33}''\cdot\mathrm{e}^{x+y}\right]$$

$$=\mathrm{e}^{x+y}f_3'-\cos x\sin yf_{12}''+\mathrm{e}^{x+y}\cos xf_{13}''-\mathrm{e}^{x+y}\sin yf_{32}''+\mathrm{e}^{2(x+y)}f_{33}'',$$

(3) $z = f(\sin x, \cos y, \mathrm{e}^{x+y})$

$$\frac{\partial z}{\partial y} = f_2'(-\sin y) + f_3'\mathrm{e}^{x+y}$$

$$= -\sin y f_2' + \mathrm{e}^{x+y} f_3',$$

$$\frac{\partial^2 z}{\partial y^2} = -\cos y f_2' - \sin y\left[f_{22}''(-\sin y) + f_{23}'' \cdot \mathrm{e}^{x+y}\right]$$

$$+ \mathrm{e}^{x+y} f_3' + \mathrm{e}^{x+y}\left[f_{32}''(-\sin y) + f_{33}'' \cdot \mathrm{e}^{x+y}\right]$$

$$= \mathrm{e}^{x+y} f_3' - \cos y f_2' + \sin^2 y f_{22}'' - 2\mathrm{e}^{x+y} \sin y f_{23}'' + \mathrm{e}^{2(x+y)} f_{33}''.$$

隐函数求导方法

(1). 利用复合函数求导法则直接计算；

(2). 利用微分形式不变性；

(3). 代公式

1)、一个方程的求导公式

方程 $F(x,y)=0$ 确定了函数 $y=y(x)$，

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F'_x}{F'_y}.$$

2). 二元隐函数的求导公式

方程 $F(x,y,z)=0$ 确定了隐函数 $z=z(x,y)$，

$$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z} \qquad \frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}$$

3)、方程组的求导公式

方程组 $\begin{cases} F(x, y, u, v)=0 \\ G(x, y, u, v)=0 \end{cases}$

雅可比行列式 $J=\dfrac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v} \\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v} \end{vmatrix} \neq 0$

$$\frac{\partial u}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}, \quad \frac{\partial u}{\partial y}=-\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}},$$

$$\frac{\partial v}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}, \quad \frac{\partial v}{\partial y}=-\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}$$

1.求下列隐函数的导数或偏导数：

(1) $\sin y + e^x - xy^2 = 0$, 求 $\dfrac{dy}{dx}$

解(1)用隐函数求导公式，

设$F(x,y)=\sin y+e^x-xy^2$,

则 $F_x = e^x - y^2$, $F_y = \cos y - 2xy$,

故 $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{e^x - y^2}{\cos y - 2xy}$

$= \dfrac{y^2 - e^x}{\cos y - 2xy}$

$$(2)\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x},\quad 求\frac{\mathrm{d}y}{\mathrm{d}x}$$

解(2) 设 $F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\dfrac{y}{x}$

$$=\frac{1}{2}\ln\left(x^2+y^2\right)-\arctan\frac{y}{x},$$

$$\because F_x=\frac{1}{2}\frac{2x}{x^2+y^2}-\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)$$

$$=\frac{x+y}{x^2+y^2},$$

$$F_y=\frac{1}{2}\frac{2y}{x^2+y^2}-\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

$$=\frac{y-x}{x^2+y^2},$$

$$\therefore\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}=\frac{x+y}{x-y}.$$

4.求由下列方程组所确定的函数的导数或偏导数：

$$(1)\begin{cases} z = x^2 + y^2, \\ x^2 + 2y^2 + 3z^2 = 20. \end{cases} \text{求 } \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}z}{\mathrm{d}x}$$

解(1)原方程组变为 $\begin{cases} y^2 - z = -x^2 \\ 2y^2 + 3z^2 = 20 - x^2 \end{cases}$

方程两边对x求导，得 $\begin{cases} 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{\mathrm{d}z}{\mathrm{d}x} = -2x \\ 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3z\dfrac{\mathrm{d}z}{\mathrm{d}x} = -x \end{cases}$

当 $J = \begin{vmatrix} 2y & -1 \\ 2y & 3z \end{vmatrix} = 6yz + 2y \neq 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{J}\begin{vmatrix}-2x & -1\\ -x & 3z\end{vmatrix}$$

$$=\frac{-6xz-x}{6yz+2y}=-\frac{x(6z+1)}{2y(3z+1)},$$

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{1}{J}\begin{vmatrix}2y & -2x\\ 2y & -x\end{vmatrix}$$

$$=\frac{2xy}{6yz+2y}=\frac{x}{3z+1}.$$