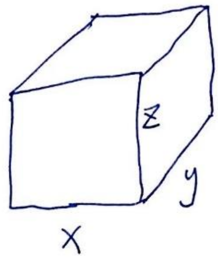
多元函数

1. 意义

$u = f(x_1, x_2, \cdots, x_n)$



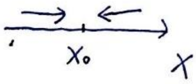
$V = x\cdot y\cdot z$

$V = f(x, y, z)$

2. 区别

① 极限

趋近路径很多



例 $\lim\limits_{(x,y)\to(1,0)} \dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}}$

① $f(x,y)$连续函数

② $(1,0)$在定义域内

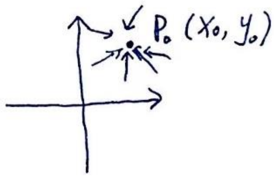
$x\to x_0^-$

$x\to x_0^+$

连续定义

$\lim\limits_{P\to P_0} f(P)=f(P_0)$

$= \dfrac{\ln(1+e^0)}{\sqrt{1+0}} = \ln 2$



$(x,y)\to(x_0,y_0)$

二重极限: 极限存在 (定义)

不存在 (反证, 反例)

$(x,y)\to(x_0,y_0)$

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=A$

$\exists\ \delta\quad 0<d((x,y),P_0)<\delta$

$|f(x,y)-A|<\varepsilon$

例2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3-\sqrt{xy+9}}{xy}$

$$= \lim \frac{(3-\sqrt{xy+9})(3+\sqrt{xy+9})}{xy(3+\sqrt{xy+9})}$$

$$= \lim \frac{9-(xy+9)}{xy(3+\sqrt{xy+9})}$$

$$= \lim_{(x,y)\to(0,0)} \boxed{-\frac{1}{3+\sqrt{xy+9}}}$$

$$= -\frac{1}{3+3} = -\frac{1}{6}$$

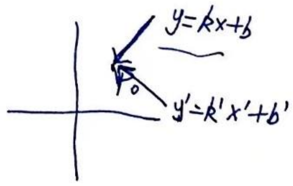
二重极限

$$\lim_{\underline{(x,y)\to(x_0,y_0)}} f(x,y) \neq \lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$$

二次极限

二重极限不同于二次极限

例3. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ 证明极限不存在



证明: $\underline{y=kx}$

$$\lim_{\substack{y=kx\\x\to0}} \frac{x^2-y^2}{x^2+y^2} = \lim \frac{x^2-(kx)^2}{x^2+(kx)^2}$$

$$= \lim_{\substack{y=kx\\x\to0}} \frac{(1-k^2)\cancel{x^2}}{(1+k^2)\cancel{x^2}}$$

$$= \frac{1-k^2}{1+k^2}$$

k取不同值,

极限不同 $\Rightarrow$ 二重极限不存在

②

$y=f(x)$

导数 $\dfrac{dy}{dx}$

微商

$u=f(x,y)$ $f(x_1,x_2)$

偏导 $\dfrac{\partial f}{\partial x}$ y视为常量

$\dfrac{\partial f}{\partial y}$ x视为常量

不是微商多

对谁偏导，谁就是变量，

剩下的变量视为常量

例3.

① $z=\dfrac{x^2+y^2}{xy}$ 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

x变量

y常量

解：化简：

$$z=\frac{x^2}{xy}+\frac{y^2}{xy}$$

$$\Rightarrow \boxed{z=\frac{x}{y}+\frac{y}{x}}$$

$$\frac{\partial z}{\partial x}=\frac{1}{y}\cdot 1+y\cdot\left(-\frac{1}{x^2}\right)$$

$$=\frac{1}{y}-\frac{y}{x^2}$$

$$\frac{\partial z}{\partial y}=x\cdot\left(-\frac{1}{y^2}\right)+\left(\frac{1}{x}\right)\cdot 1$$

$$=-\frac{x}{y^2}+\frac{1}{x}$$

② $u=\left(\frac{y}{x}\right)^z$

求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$

解: $\boxed{\frac{\partial u}{\partial x}=z\cdot\left(\frac{y}{x}\right)^{z-1}\cdot y\cdot\left(-\frac{1}{x^2}\right)}=-\frac{y}{x^2}\cdot z\cdot\frac{y^{z-1}}{x^{z-1}}$

$\left(\frac{y}{x}\right)^z$ $\qquad (x^n)'=n\cdot x^{n-1}$ $\qquad =-\frac{z\cdot y^z}{x^{z+1}}$

$\left(\frac{y}{x}\right)'$

$\boxed{\frac{\partial u}{\partial y}=z\cdot\left(\frac{y}{x}\right)^{z-1}\cdot\frac{1}{x}\cdot 1}=\frac{z\cdot y^{z-1}}{x^z}$

$\boxed{\frac{\partial u}{\partial z}=\left(\frac{y}{x}\right)^z\cdot\ln\left(\frac{y}{x}\right)}$

$\left(\frac{y}{x}\right)^z$ $\qquad (a^x)'=a^x\cdot\ln a$

常量 $\qquad (e^x)'=e^x$

$\left(\frac{y}{x}\right)^z$ $\qquad (x^n)'=n\cdot x^{n-1}$

$(a^x)'=a^x\cdot\ln a.$

$$\vec{g} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \quad \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

梯度

f_1 f_2

位置

$f(\underset{=}{x}, \underset{=}{y})$

$f(x_1, x_2, \cdots, x_n)$

$f_1 \quad f_2 \cdots f_n$

$f(\underbrace{tx+a}_{u}, \; ny+b)$

$$f_1 = \frac{\partial f}{\partial u} \cdot \frac{du}{dx}$$

$$f_x(x,y) = \frac{\partial f}{\partial x}$$

$$f_y(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x} = \underline{\frac{\partial^2 f}{\partial x^2}} = \underline{f_{xx}}(x,y)$$

$$\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y} = \underline{\frac{\partial^2 f}{\partial x \cdot \partial y}} = \underline{f_{xy}}(x,y)$$

① f_{xy} 与 f_{yx} 存在且连续

$\Longleftrightarrow f_{xy} = f_{yx}$

$$\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x} = \underline{\frac{\partial^2 f}{\partial y \cdot \partial x}} = \underline{f_{yx}}(x,y)$$

$$\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y} = \frac{\partial^2 f}{\partial y \cdot \partial y} = \underline{\frac{\partial^2 f}{\partial y^2}} = \underline{f_{yy}}(x,y)$$