Degree Examination

JC1004 Advanced Mathematics I-1

2022/1/14 (09:00–11:00)

Calculators are not permitted in this examination.

Marks may be deducted for answers that do not show clearly how the solution is reached. Attempt ALL FIVE questions from SECTION A and TWO questions from SECTION B.

All questions are worth 10 marks.

SECTION A — answer all FIVE questions

1. **(10 marks)** Compute the following limits.

(a) $\lim_{x\to\frac{\pi}{4}}(x\tan x - 1)$ **(b)** $\lim_{x\to -1}\dfrac{x^2+3x+2}{x+1}$ **(c)** $\lim_{x\to 0}\dfrac{\sin x^3}{\sin^2 x}$

(d) $\lim_{x\to -6}\dfrac{|x|-6}{x+6}$ **(e)** $\lim_{x\to 0}\dfrac{\sqrt{a^2+x}-a}{x}$ $(a>0)$

2. **(a)** **(6 marks)** Differentiate the functions f, g and h defined as follows.

$f(x) = x\sqrt{1-x^2}$ $g(x) = \ln(\sin x)$ $h(x) = \int_3^{2x} e^{-t^2}dt$

(b) **(4 marks)** Calculate the indefinite integrals.

$\int \dfrac{4x^3+6}{x^4}dx$ $\int \dfrac{x}{e^{6x}}dx$

3. **(a)** **(3 marks)** Use the precise definition of the limit to show that $\lim_{x\to 3}(x^2-2x+2) = 5$.

(b) **(4 marks)** Use implicit differentiation to find an equation of the tangent line to the curve

$$y\sin 2x = x\cos 2y$$

at the given point $(\pi/2, \pi/4)$.

(c) **(3 marks)** Compute the area of the region enclosed by the curves $y^2 = x$ and $x-2y-3 = 0$.

4. **(a)** **(2 marks)** Let $f(x) = \dfrac{\tan x}{1+x^2+x^4}$. State whether the function f is even, odd, both even and odd, or neither even nor odd.

(b) **(4 marks)** Find the critical numbers of $f(x) = x^{3/5}(x-3)^3$.

(c) **(4 marks)** Let f be a twice differentiable function such that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Compute $\int \left(\dfrac{f(x)}{f'(x)} - \dfrac{f^2(x)f''(x)}{(f'(x))^3}\right)dx$.

5. Let f be the function defined by $f(x) = x^3 + 6x^2 - 15x$.

(a) (4 marks) Use the definition of the derivative in terms of a limit to show that $f'(x) = 3x^2 + 12x - 15$.

(b) (3 marks) On what intervals is f increasing? On what intervals is f decreasing?

(c) (3 marks) Show that $f(x)$ has exactly one real root in the interval $(1, 2)$.

SECTION B — answer TWO questions

6. **(a) (4 marks)** Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

(b) (6 marks) Find $h'(2)$, given that $f(2) = -3, g(2) = 4, f'(2) = -2$ and $g'(2) = 7$.

$$h(x) = f(x)g(x) \qquad h(x) = \frac{f(x)}{g(x)} \qquad h(x) = \frac{g(x)}{1 + f(x)}$$

7. **(a) (4 marks)** A number c is called a **fixed point** of a function if $f(c) = c$. Prove that if for all real numbers x, we have $f'(x) > 1$, then f has at most one fixed point.

(b) (6 marks) Let f be a differentiable even function with $f(3) = 2$, and $f(-1) = -1$, and $f'(x) > 0$ for $x > 0$.

(i) Sketch a possible graph for f.

(ii) How many solutions does the equation $f(x) = 0$ have? Why?

8. **(a) (6 marks)** Let $f(x)$ be a function and $F(x)$ an antiderivative of $f(x)$. Suppose that $f(x)F(x) = \sin^2 2x$ for $x \geqslant 0$ and $F(0) = 1$, $F(x) \geqslant 0$. Find $f(x)$.

(b) (4 marks) Evaluate the integral.

$$\int_2^5 |x - 3| dx \qquad \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x + 1} dx$$

UNIVERSITY OF ABERDEEN **SESSION 2021–22**

DEGREE EXAMINATION SOLUTIONS

JC1004 Advanced Mathematics I-1

2022/1/14 (09:00–11:00)

SECTION A — answer all FIVE questions

1. (2 marks per part.)

(a) $\lim_{x\to\frac{\pi}{4}}(x\tan x - 1) = \frac{\pi}{4}\tan\frac{\pi}{4} - 1 = \frac{\pi}{4} - 1$

(b) $\lim_{x\to -1}\frac{x^2+3x+2}{x+1} = \lim_{x\to -1}\frac{(x+1)(x+2)}{x+1} = \lim_{x\to -1}(x+2) = 1$

(c) $\lim_{x\to 0}\frac{\sin x^3}{\sin^2 x} = \lim_{x\to 0}\left(\frac{\sin x^3}{x^3}\frac{x}{\frac{\sin^2 x}{x^2}}\right) = \lim_{x\to 0}\frac{\sin x^3}{x^3}\lim_{x\to 0}\frac{x}{\frac{\sin^2 x}{x^2}} = 1\times\frac{0}{1} = 0$

(d) $\lim_{x\to -6}\frac{|x|-6}{x+6} = \lim_{x\to -6}\frac{-x-6}{x+6} = -1$

(e) $\lim_{x\to 0}\frac{\sqrt{a^2+x}-a}{x} = \lim_{x\to 0}\frac{(\sqrt{a^2+x}-a)(\sqrt{a^2+x}+a)}{(\sqrt{a^2+x}+a)x} = \lim_{x\to 0}\frac{a^2+x-a^2}{(\sqrt{a^2+x}+a)x} = \lim_{x\to 0}\frac{x}{(\sqrt{a^2+x}+a)x} = \lim_{x\to 0}\frac{1}{\sqrt{a^2+x}+a} = \frac{1}{2a}$

2. (a) (2 marks per part.)

(1) $f'(x) = \sqrt{1-x^2} + x\frac{-2x}{2\sqrt{1-x^2}} = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$

(2) $g'(x) = \frac{1}{\sin x}\cos x = \cot x$

(3) $h'(x) = e^{-(2x)^2}(2x)' = 2e^{-4x^2}$

(b) (2 marks per part.)

(1) $\int\frac{4x^3+6}{x^4}dx = \int 4x^{-1} + 6x^{-4}dx = 4\ln|x| + \frac{6}{-3}x^{-3} + C = 4\ln|x| - 2x^{-3} + C.$

(2) Use

$$u = x, dv = e^{-6x}dx$$

$$du = dx, v = -\frac{1}{6}e^{-6x}$$

to get

$$\int xe^{-6x}dx = \int udv = uv - \int vdu(IBP) = -\frac{1}{6}\left(xe^{-6x} - \int e^{-6x}dx\right) = -\frac{1}{6}xe^{-6x} - \frac{1}{36}e^{-6x} + C$$

3. (a) (3 marks) Let $\epsilon > 0$. Choose $\delta = \min\{1, \frac{\epsilon}{5}\}$. Suppose that $0 < |x-3| < \delta$.

Then since $\delta \leqslant 1$ and $|x-3| < \delta$, we have $|x-3| < 1$. That means $-1 < x-3 < 1$, so by adding 4 to every term we see that $3 < x+1 < 5$, so that $|x+1| < 5$.

Thus $|(x^2-2x+2)-5| = |x^2-2x-3| = |(x-3)(x+1)| < 5\delta < \epsilon$.

(b) (4 marks) Since $y\sin 2x = x\cos 2y$, differentiating both sides gives $y'\sin 2x + 2y\cos 2x = \cos 2y - 2xy'\sin 2y$, which rearranges to $y' = \frac{\cos 2y - 2y\cos 2x}{\sin 2x + 2x\sin 2y}$.

For $x = \pi/2, y = \pi/4$, this gives $y' = \frac{\cos\frac{\pi}{2} - \frac{\pi}{2}\cos\pi}{\sin\pi + \pi\sin\frac{\pi}{2}} = \frac{\pi/2}{\pi} = \frac{1}{2}$. So the equation of the tangent is $y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$, i.e.,

$$y = \frac{1}{2}x.$$

(c) (3 marks) Substituting $x = y^2$ into the second equation gives $y^2 - 2y - 3 = (y-3)(y+1) = 0$. The intersection points are therefore $(1, -1)$ and $(9, 3)$. The area can be computed as

$$Area = \int_0^1 (\sqrt{x} - (-\sqrt{x}))dx + \int_1^9 \left(\sqrt{x} - \frac{x-3}{2}\right) dx = 2\int_0^1 \sqrt{x}dx + \int_1^9 \left(\sqrt{x} - \frac{x-3}{2}\right) dx = \frac{4}{3} + \frac{28}{3} = \frac{32}{3}$$

4. (a) (2 marks) Since $f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4} = \frac{-\tan x}{1+x^2+x^4} = -f(x)$, then $f(x)$ is odd.

(b) (4 marks) Since $f'(x) = \frac{3}{5}(x-3)^3 x^{-\frac{2}{5}} + 3x^{\frac{3}{5}}(x-3)^2 = \frac{3}{5}(x-3)^2((x-3)x^{-\frac{2}{5}} + 5x^{\frac{3}{5}}) = \frac{9}{5}(x-3)^2 x^{-\frac{2}{5}}(2x-1)$, the critical numbers are 0, where $f'(x)$ does not exist, and 3 and 1/2, where $f'(x) = 0$.

(c) (4 marks) $\int \left(\frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{(f'(x))^3}\right) dx = \int \frac{f(x)}{f'(x)} \left(1 - \frac{f(x)f''(x)}{(f'(x))^2}\right) dx = \int \frac{f(x)}{f'(x)} \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} dx = \int \frac{f(x)}{f'(x)} d\left(\frac{f(x)}{f'(x)}\right) = \frac{1}{2}\left(\frac{f(x)}{f'(x)}\right)^2 + C$

5. (a) (4 marks)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 + 6(x+h)^2 - 15(x+h) - (x^3 + 6x^2 - 15x)}{h} \\ &= \lim_{h\to 0} \frac{h^3 + 3x^2h + 3xh^2 + 6h^2 + 12xh - 15h}{h} \\ &= \lim_{h\to 0} (h^2 + 3x^2 + 3xh + 6h + 12x - 15) \\ &= 3x^2 + 12x - 15 \end{aligned}$$

Since $f'(x) = 3x^2 + 12x - 15 = 3(x^2 + 4x - 5) = 3(x+5)(x-1)$ is defined for all x, then the critical numbers are just the c for which $f'(c) = 0$, or in other words, $c = -5, 1$.

(b) (3 marks) On $(-\infty, -5)$ we have $x+5 < 0$, $x-1 < 0$, so that $f'(x) > 0$. Consequently f is increasing on $(-\infty, -5)$.

On $(-5, 1)$ we have $x+5 > 0$, $x-1 < 0$, so that $f'(x) < 0$. Consequently f is decreasing on $(-5, 1)$.

On $(1, \infty)$ we have $x+5 > 0$, $x-1 > 0$, so that $f'(x) > 0$. Consequently f is increasing on $(1, \infty)$.

(c) (3 marks) Note that $f(1) = -8, f(2) = 2$, and f is continuous on the interval $[1, 2]$. So by intermediate value theorem, there is $c \in (1, 2)$ such that $f(c) = 0$, i.e., $f(x)$ has one real root in the interval $(1, 2)$. Suppose that this equation has a second root c'. Then since $f(c) = f(c') = 0$ and f is differentiable everywhere, Rolle's theorem shows that there is $x_0 \in (c, c')$ such that $f(x_0) = 0$. However $f'(x) > 0$ for all $x \in (1, 2)$ so that no such x_0 exists, and so no second root could have existed.

SECTION B — answer TWO questions

6. (a) (4 marks) By the fact that rational functions and polynomials are continuous at every point of their domains, the function is clearly continuous at every real number except perhaps 2, 3. In order for f to be continuous at 2, we need that $f(2)$ is defined, that $\lim_{x\to 2} f(x)$ exists, and that $\lim_{x\to 2} f(x) = f(2)$. Certainly $f(2)$ is defined; the limit exists if and only if

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

or in other words

$$\lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^+} ax^2 - bx + 3$$

or in other words

$$4 = 4a - 2b + 3$$

and in this case the third condition holds.

In order for f to be continuous at 3, we need that $f(3)$ is defined, that $\lim_{x\to 3} f(x)$ exists, and that $\lim_{x\to 3} f(x) = f(3)$. Certainly $f(3)$ is defined; the limit exists if and only if

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

or in other words

$$\lim_{x\to 3^-} ax^2 - bx + 3 = \lim_{x\to 3^+} 2x - a + b$$

or in other words

$$9a - 3b + 3 = 6 - a + b.$$

Thus $a = b = \frac{1}{2}$.

(b) (2 marks per part.)

$h(2) = f'(2)g(2) + f(2)g'(2) = -2 \times 4 + (-3) \times 7 = -29$,

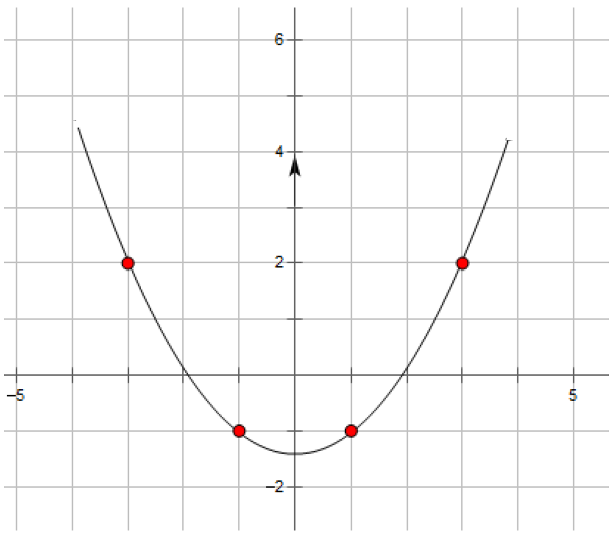
$h(2) = \frac{f'(2)g(2) - f(2)g'(2)}{g^2(2)} = \frac{(-2)\times 4 - (-3)\times 7}{16} = \frac{13}{16}$,

$h(2) = \frac{g'(2)(1+f(x)) - g(2)f'(2)}{(1+f(2))^2} = \frac{7\times(1-3) - (-2)\times 4}{4} = -\frac{3}{2}$.

7. **(a) (4 marks)** Since $f'(x)$ exists for all x, then f is continuous. Let $g(x) = f(x) - x$. Then g is continuous and $g'(x) = f'(x) - 1 > 0$ follows from $f'(x) > 1$. Thus g is an increasing function and then $g(x) = 0$ has at most one root, in other words, f has at most one fixed point.

(b) (3 marks per part.)

(i)



(ii) The equation $f(x) = 0$ has two solutions. By the fact that f is an even function and $f(-1) = -1$, we have $f(1) = -1$. Hence, by f is differentiable, we have f is continuous. Thus, by $f(3) = 2$, $f(1) = -1$ and intermediate value theorem, there exists one root for $x \in [1, 3]$. Since $f'(x) > 0$ for $x > 0$, there is no root for $x \in [0, 1]$ and $x \in (3, +\infty)$. Similarly, by the fact that f is an even function, there exists only root for $x \in (-\infty, 0)$. Thus, there are two roots for the function f.

8. **(a) (6 marks)** Since $F'(x) = f(x)$, we have $F'(x)F(x) = f(x)F(x) = \sin^2(2x)$. So $[F^2(x)]' = 2F'(x)F(x) = 2\sin^2(2x)$ and then $F^2(x) = \int 2\sin^2(2x)dx = \int(1-\cos(4x))dx = x - \frac{1}{4}\sin(4x) + c$. Note that $F(0) = 1$. Thus $c = 1$ and $F^2(x) = x - \frac{1}{4}\sin(4x) + 1$. Now since $F(x) \geqslant 0$, $F(x) = \sqrt{x - \frac{1}{4}\sin(4x) + 1}$ follows. Consequently,

$$f(x) = F'(x) = \frac{1}{2}\frac{1-\cos(4x)}{\sqrt{x - \frac{1}{4}\sin(4x) + 1}} = \frac{\sin^2(2x)}{\sqrt{x - \frac{1}{4}\sin(4x) + 1}}.$$

(b) (2 marks per part.)

(1)

$$\int_2^5 |x-3| = \int_2^3 (3-x)dx + \int_3^5 (x-3)dx = (3x - \frac{1}{2}x^2)|_2^3 + (\frac{1}{2}x^2 - 3x)|_3^5 = \frac{5}{2}.$$

(2) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x + 1}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x + 1}d(\sin x)$. Using $\sin x = t$, we get

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x + 1}dx = \int_0^1 \frac{1}{t^2+1}dt = \arctan t|_0^1 = \frac{\pi}{4}.$$

UNIVERSITY OF ABERDEEN **SESSION 2021–22**

DEGREE EXAMINATION SETTER'S COMMENTS

JC1004 Advanced Mathematics I-1

2022/1/14 (09:00–11:00)

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0. All questions are standard book work, and multiple examples of each type have appeared on example sheets.

UNIVERSITY OF ABERDEEN **SESSION 2021–22**

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Cover Page

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External	

Please note below any Tables or other documents to be provided during the exam.

Before paper goes to External Examiner.....

Please make sure that you have provided *please tick boxes*

- Solutions, with provisional marking scheme ☐
- Setter's comments (and any comments that the vetter wishes to include) ☐
- CA information to date (if appropriate) ☐

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