101. (a): Expanding gives

$$\begin{aligned}\int (2t-7)^3\,dt &= \int (2t)^3 + 3\cdot(-7)(2t)^2 + 3\cdot(-7)^2(2t) + (-7)^3\,dt \\ &= \int 8t^3 - 84t^2 + 294t - 343\,dt \\ &= 8\tfrac{t^4}{4} - 84\tfrac{t^3}{3} + 294\tfrac{t^2}{2} - 343t + C \\ &= 2t^4 - 28t^3 + 147t^2 - 343t + C.\end{aligned}$$

(b): Use

$$u = 2t-7, \quad du = 2dt$$

to get

$$\begin{aligned}\int (2t-7)^3\,dt &= \int u^3\,\frac{dt}{2} \\ &= \frac{u^4}{8} \\ &= \frac{(2t-7)^4}{8} + C.\end{aligned}$$

Note that the two answers differ by a constant.

102. (a): Using the angle-sum trig identities, we get

$$\begin{aligned}\sin(x)^3 &= \sin(x)\sin(x)^2\\ &= \sin(x)\frac{1-\cos(2x)}{2}\\ &= \frac{1}{2}(\sin(x)-\sin(x)\cos(2x))\\ &= \frac{1}{2}(\sin(x)-\frac{1}{2}(\sin(x+2x)+\sin(x-2x)))\\ &= \frac{1}{2}\sin(x)-\frac{1}{4}\sin(3x)+\frac{1}{4}\sin(x)\\ &= \frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x),\end{aligned}$$

and therefore

$$\begin{aligned}\int \sin(x)^3\,dx &= \int \frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x)\,dx\\ &= -\frac{3}{4}\cos(x)+\frac{1}{12}\cos(3x)+C.\end{aligned}$$

(b): Use

$$u=\cos(x), \quad du=-\sin(x)\,dx$$

to get

$$\begin{aligned}\int \sin(x)^3\,dx &= \int (1-\cos(x)^2))\sin(x)\,dx\\ &= -\int (1-u^2)\,du\\ &= -u+\frac{u^3}{3}+C\\ &= -\cos(x)+\frac{\cos(x)^3}{3}+C.\end{aligned}$$

These answers might look very different; however, they can be shown to be equal using more trig double-angle identities.

103. (a): Using

$$u=\cos(t), \quad du=-\sin(t)\,dt,$$

we get

$$\begin{aligned}\int \sin(t)\left(\cos(t)^4+6\cos(t)^3-6\right)\,dt &= -\int (u^4+6u^3-6)\,du\\ &= -\frac{u^5}{5}+\frac{6u^4}{4}+6u+C\\ &= -\frac{\cos(t)^5}{5}+\frac{3\cos(t)^4}{2}+6\cos(t)+C.\end{aligned}$$

(b): Using

$$u = \ln(x), \quad du = \frac{1}{x}\,dx,$$

we get

$$\begin{aligned}\int \frac{1}{x}(\ln(x) - \ln(x)^3 + 2\cos(\ln(x)))\,dx &= \int u - u^3 + 2\cos(u)\,du \\ &= \frac{u^2}{2} - \frac{u^4}{4} + 2\sin(u) \\ &= \frac{\ln(x)^2}{2} - \frac{\ln(x)^4}{4} + 2\sin(\ln(x)).\end{aligned}$$

(c): Note that $w^w = e^{w\ln(w)}$. Using

$$u = w\ln(w), \quad du = w\frac{1}{w} + \ln(w)\,dw = 1 + \ln(w)\,dw,$$

we get

$$\begin{aligned}\int w^w(1 + \ln(w))\,dw &= \int e^u\,du \\ &= e^u + C \\ &= e^{w\ln(w)} + C \\ &= w^w + C.\end{aligned}$$

It is also possible to do this one by substituting $u = w^w$.

(d): Using

$$u = 3x^2 + 5, \quad du = 6x\,dx,$$

we get

$$\int \frac{x\cos(3x^2+5)}{\sin(3x^2+5)+6}\,dx = \int \frac{\cos(u)}{6(\sin(u)+6)}\,du.$$

We need a further substitution. Use

$$v = \sin(u) + 6, \quad dv = \cos(u)\,du$$

to get

$$\begin{aligned}\int \frac{x\cos(3x^2+5)}{\sin(3x^2+5)+6}\,dt &= \int \frac{\cos(u)}{6(\sin(u)+6)}\,du \\ &= \int \frac{1}{6v}\,dv \\ &= \frac{\ln|v|}{6} + C \\ &= \frac{\ln|(\sin(u)+6)|}{6} + C \\ &= \frac{\ln|(\sin(3x^2+5)+6)|}{6} + C.\end{aligned}$$

One could do this with one substitution, namely $u = \sin(3x^2 + 5) + 6$.

(e): Using

$$u = \sqrt{\tan(x) + 17}, \quad du = \frac{1}{2}\sec(x)^2(\tan(x) + 17)^{-1/2}\,dx,$$

we get

$$\begin{aligned}\int \sec(x)^2 \frac{e^{\sqrt{\tan(x)+17}}}{\sqrt{\tan(x) + 17}}\,dx \\ &= \int e^u\,2du \\ &= 2e^u + C \\ &= 2e^{\sqrt{\tan(x)+17}} + C.\end{aligned}$$

104. (a): Note that

$$\frac{A}{B + Cx} = A\frac{1}{B + Cx}.$$

It therefore seems that we should substitute

$$u = B + Cx, \quad du = C, dx,$$

which gives

$$\begin{aligned}\int \frac{A}{B + Cx}\,dx &= A\int \frac{1}{u}\frac{1}{C}du \\ &= \frac{A}{C}\ln|u| + K \\ &= \frac{A}{C}\ln|B + Cx| + K.\end{aligned}$$

(b): Note that

$$\frac{A}{B + Cx^2} = \frac{A}{B}\frac{1}{1 + (C/B)x^2} = \frac{A}{B}\frac{1}{1 + (\sqrt{(}C/B)x)^2}.$$

Therefore it makes sense to substitute

$$u = \sqrt{\frac{C}{B}}x, \quad du = \sqrt{\frac{C}{B}}\,dx,$$

which gives

$$\begin{aligned}\int \frac{A}{B + Cx^2}\,dx &= \frac{A}{B}\int \frac{1}{1 + u^2}\sqrt{\frac{B}{C}}\,du \\ &= \frac{A\sqrt{B}}{B\sqrt{C}}\tan^{-1}(u) + K \\ &= \frac{A}{\sqrt{BC}}\tan^{-1}\left(\sqrt{\frac{C}{B}}x\right) + K.\end{aligned}$$

(c) Note that

$$\frac{A}{\sqrt{B-Cx^2}} = \frac{A}{\sqrt{B}}\frac{1}{\sqrt{1-(\sqrt{C/B}x)^2}},$$

and so we substitute

$$u = \sqrt{\frac{C}{B}}x, \quad du = \sqrt{\frac{C}{B}}\,dx.$$

This leads to

$$\begin{aligned}\int \frac{A}{\sqrt{B-Cx^2}}\,dx &= \frac{A}{\sqrt{B}}\int \frac{1}{\sqrt{1-u^2}}\sqrt{\frac{B}{C}}\,du \\ &= \frac{A}{\sqrt{C}}\sin^{-1}(u)+K \\ &= \frac{A}{\sqrt{C}}\sin^{-1}\left(\sqrt{\frac{C}{B}}x\right)+K.\end{aligned}$$

105. (a): Since $\rho'(x) = e^{-x^2}$, we have (by definition)

$$\int \frac{2e^{-x^2}}{\sqrt{\pi}}\,dx = \frac{2\rho(x)}{\sqrt{\pi}} + C.$$

(b): Note that

$$xe^{-1+2x^2-x^4} = xe^{-(1-x^2)^2},$$

which suggests the substitution

$$u = 1-x^2, \quad du = -2x\,dx.$$

With this, we get

$$\begin{aligned}\int xe^{-1+2x^2-x^4}\,dx &= -\frac{1}{2}\int e^{-u^2}\,du \\ &= -\frac{1}{2}\rho(u) \\ &= -\frac{1}{2}\rho(1-x^2).\end{aligned}$$

(c): First use the substitution

$$u = t^3, \quad du = 3t^2\,dt$$

to get

$$\int t^2\rho(t^3)e^{-t^6}\,dt = \frac{1}{3}\int \rho(u)e^{-u^2}\,du.$$

Now use the further substitution

$$v = \rho(u), \quad dv = e^{-u^2}\,du$$

to get

$$\begin{aligned}\int t^2\rho(t^3)e^{-t^6}\,dt &= \frac{1}{3}\int \rho(u)e^{-u^2}\,du\\ &= \frac{1}{3}\int v\,dv\\ &= \frac{1}{6}v^2 + C\\ &= \frac{1}{6}\rho(u)^2 + C\\ &= \frac{1}{6}\rho(t^3)^3 + C.\end{aligned}$$

106. (a): This area is given by the integral

$$\begin{aligned}\int_0^\pi \sin(x)\,dx &= -\cos(x)\big|_{x=0}^{\pi}\\ &= -\cos(\pi) - \cos(0) \qquad\qquad = 1 - 1 = 2.\end{aligned}$$

(b): The two curves intersect when

$$x^3 - 25x = -x^3 - x + 32$$

which can be rearranged as

$$\begin{aligned}0 &= 2x^3 - 24x - 32\\ &= 2(x-4)(x+2)^2.\end{aligned}$$

Thus, the points of intersection are

$$(-2, (-2)^3 - 25(-2)) = (-2, 42)$$

and

$$(4, 4^3 - 25\cdot 4) = (4, -36).$$

Note (e.g., by evaluating at 0) that the curve $y = x^3 - 25x$ lies below the curve $y = -x^3 - x - 32$ for $x \in (-2, 4)$.

As in Example 1.5.6, this area is given by

$$\begin{aligned}\int_{-2}^{4} (-x^3 - x + 32) - (x^3 - 25x)\,dx &= \int_{-2}^{4} -2x^3 + 24x + 32\,dx\\ &= -\frac{2x^4}{4} + \frac{24x^2}{2} + 32x\bigg|_{x=-2}^{4}\\ &= -\frac{4^4}{2} + 12(4) + 32(4) - \left(-\frac{(-2)^4}{2} + 12(-2)^2 + 32(-2)\right)\\ &= 216.\end{aligned}$$

(c): As in Example 1.5.7 (b), we integrate over y instead of x, since it is very difficult to express the first curve in the form $y = f(x)$. The area is then given by

$$\int_{\ln(\pi)}^{\ln(2\pi)} -e^y \sin(e^y) - (-y)\,dy = \int_{\ln(\pi)}^{\ln(2\pi)} -e^y \sin(e^y)\,dy + \int_{\ln(\pi)}^{\ln(2\pi)} y\,dy$$

For the first of these integrals, we want to substitute

$$u = e^y, \quad du = e^y\,dy$$

to get

$$\begin{aligned}\int e^y \sin(e^y)\,dy &= \int \sin(u)\,du\\ &= -\cos(u) + C\\ &= -\cos(e^y) + C.\end{aligned}$$

Hence, the area is

$$\begin{aligned}\int_{\ln(\pi)}^{\ln(2\pi)} e^y \sin(e^y) - (-y)\,dy &= \cos(e^y) + \frac{y^2}{2}\Big|_{y=\ln(\pi)}^{\ln(2\pi)}\\ &= \cos(2\pi) + \frac{\ln(2\pi)^2}{2} - \left(\cos(\pi) + \frac{\ln(\pi)^2}{2}\right)\\ &= 1 + \frac{(\ln(2)+\ln(\pi))^2}{2} - (-1) - \frac{\ln(\pi)^2}{2}\\ &= 2 + \frac{\ln(2)^2}{2} + \ln(2)\ln(\pi) + \frac{\ln(\pi)^2}{2} - \frac{\ln(\pi)^2}{2}\\ &= 2 + \frac{\ln(2)^2}{2} + \ln(2)\ln(\pi).\end{aligned}$$

(d): It is very difficult to write the first curve in the form $y = f(x)$ or (equally) the second curve in the form $x = f(y)$. So, this question requires a slightly different approach than what we did before.

We break this region into two pieces along the line $y = x$. We can see that each piece is a reflection of the other, so the total area is

$$2\int_0^1 \frac{x^2\sqrt{10-x^3}}{3}\,dx.$$

To solve this integral, we use the substitution

$$u = 10 - x^3, \quad du = -3x^2\,dx.$$

This gives

$$\begin{aligned}\int \frac{x^2\sqrt{10-x^3}}{3}\,dx &= -\frac{1}{9}\int \sqrt{u}\,du\\ &= -\frac{1}{9}\frac{u^{3/2}}{3/2} + C\\ &= -\frac{2}{27}(10-x^3)^{3/2} + C.\end{aligned}$$

and therefore, the area is

$$2\int_0^1 \frac{x^2\sqrt{10-x^3}}{3}\,dx = -\frac{4}{27}(10-x^3)^{3/2}\Big|_{x=0}^{1}$$
$$= -\frac{4}{27}\left(9^{3/2}-10^{3/2}\right)$$
$$= -4+\frac{40\sqrt{10}}{27}.$$

107. (a): By FTC1 and the Chain Rule (as in Example 1.5.3 (ii)), we have

$$f'(x) = e^{-(2x)^2}\cdot\left(\frac{d}{dx}2x\right) = 2e^{-4x^2}.$$

(b): To do this, first rewrite it as a difference of integrals:

$$g(x) = \int_{-x}^{0}\sqrt[3]{\cos(t)}\,dt + \int_0^{x^2}\sqrt[3]{\cos(t)}\,dt$$
$$= -\int_0^{-x}\sqrt[3]{\cos(t)}\,dt + \int_0^{x^2}\sqrt[3]{\cos(t)}\,dt.$$

Now we differentiate each term using FTC1 and the Chain Rule, as before.

$$g'(x) = \sqrt[3]{\cos(-x)} + 2x\sqrt[3]{\cos(x^2)}.$$

(c): This one is a bit different, because the integrand contains x. However, we can factor x out of the integrand, then use the Product Rule.

$$h(x) = \int_x^0 -x^2\sqrt{1+t^3}\,dt$$
$$= x^2\int_0^x\sqrt{1+t^3}\,dt$$
$$h'(x) = 2x\int_0^x\sqrt{1+t^3}\,dt + x^2\sqrt{1+t^3}.$$

It is alright to leave the answer like this (with a definite integral).

108. (a): Using the answer to Question 4(b) on Exercise Sheet 1, we have

$$\int_0^{5/2}\frac{1}{4x^2+25}\,dx = \frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right)\Big|_{x=0}^{5/2}$$
$$= \frac{1}{10}\left(\tan^{-1}(1)-\tan^{-1}(0)\right)$$
$$= \frac{1}{10}\left(\frac{\pi}{4}-0\right)$$
$$= \frac{\pi}{40}.$$

(b): Use the substitution

$$u = 4x^2 + 25, \quad du = 8x\,dx$$

to get

$$\begin{aligned}\int \frac{x}{4x^2+25}\,dx &= \int \frac{1}{8u}\,du \\ &= \frac{\ln(u)}{8} + C \\ &= \frac{\ln(4x^2+25)}{8} + C.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{5/2} \frac{x}{4x^2+25}\,dx &= \left.\frac{\ln(4x^2+25)}{8}\right|_{x=0}^{5/2} \\ &= \frac{1}{8}(\ln(25+25) - \ln(25)) \\ &= \frac{\ln(25)+\ln(2)-\ln(25)}{8} \\ &= \frac{\ln(2)}{8}.\end{aligned}$$

(c): Using the substitution

$$u = x^3 - 2x^{3/2} + 2, \quad du = 3x^2 - 2\cdot\frac{3}{2}\cdot\sqrt{x}\,dx,$$

we get

$$\begin{aligned}\int \frac{x^2-\sqrt{x}}{(x^3-2x^{3/2}+2)^2}\,dx &= \frac{1}{3}\int \frac{1}{u^2}\,du \\ &= -\frac{1}{3u} + C \\ &= -\frac{1}{3(x^3-2x^{3/2}+2)} + C.\end{aligned}$$

Hence,

$$\begin{aligned}\int_1^4 \frac{x^2-\sqrt{x}}{(x^3-2x^{3/2}+2)^2}\,dx &= -\left.\frac{1}{3(x^3-2x^{3/2}+2)}\right|_{x=1}^{4} \\ &= -\frac{1}{3}\left(-\frac{1}{64-16+2} - \frac{1}{1-2+2}\right) \\ &= -\frac{1}{3}\cdot\frac{1-50}{50} \\ &= \frac{49}{150}.\end{aligned}$$

(d): First, use the substitution

$$u = 1 - \cos(x)^2, \quad du = 2\cos(x)\sin(x)\,dx$$

to get

$$\int \sin(x)\cdot\cos(x)\cdot\cos\left(\frac{\pi}{e-1}\left(e^{1-\cos(x)^2}-1\right)\right)\cdot e^{1-\cos(x)^2}\,dx$$
$$= \int \cos\left(\frac{\pi}{e-1}\left(e^u-1\right)\right)\cdot e^u\,du.$$

Next substitute

$$v = \frac{\pi(e^u-1)}{e-1}, \quad dv = \frac{\pi(e^u)}{e-1}\,du$$

to get

$$\begin{aligned}
&\int \sin(x)\cdot\cos(x)\cdot\cos\left(\frac{\pi}{e-1}\left(e^{1-\cos(x)^2}-1\right)\right)\cdot e^{1-\cos(x)^2}\,dx\\
&= \int \cos\left(\frac{\pi}{e-1}\left(e^u-1\right)\right)\cdot e^u\,du\\
&= \frac{e-1}{\pi}\int \cos\left(v\right)\,dv\\
&= \frac{e-1}{\pi}\sin(v)\\
&= \frac{e-1}{\pi}\sin\left(\frac{\pi(e^u-1)}{e-1}\right)\\
&= \frac{e-1}{\pi}\sin\left(\frac{\pi(e^{1-\cos(x)^2}-1)}{e-1}\right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_0^{\pi/2} \sin(x)\cdot\cos(x)\cdot\cos\left(\frac{\pi}{e-1}\left(1-e^{\cos(x)^2}-1\right)\right)\cdot e^{1-\cos(x)^2}\,dx\\
&= \frac{e-1}{\pi}\sin\left(\frac{\pi(e^{1-\cos(x)^2}-1)}{e-1}\right)\Bigg|_{x=0}^{\pi/2}\\
&= \frac{e-1}{\pi}\left(\sin\left(\frac{\pi(e^{1-\cos(\pi/2)^2}-1)}{e-1}\right)-\sin\left(\frac{\pi(e^{1-\cos(0)^2}-1)}{e-1}\right)\right)\\
&= \frac{e-1}{\pi}\left(\sin\left(\frac{\pi(e-1)}{e-1}\right)-\sin\left(\frac{\pi(1-1)}{e-1}\right)\right)\\
&= \frac{e-1}{\pi}(-1-1)\\
&= \frac{2(1-e)}{\pi}.
\end{aligned}$$