# Limit

## 1. Basic Knowledge of Limit

### 1.1 the concept of limit

① $\lim_{x\to a} f(x) = L$ $\qquad \lim_{x\to a} f(x) = \infty$

② $\lim_{x\to \pm\infty} f(x) = L$ $\qquad \lim_{x\to\infty} f(x) = \infty$

① $\Leftrightarrow$ $\forall \varepsilon > 0$. $\exists \delta > 0$. s.t if $|x-a| < \delta$
then $|f(x) - L| < \varepsilon$ $\quad < <$ $\quad < >$

② $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists k > 0$ s.t if $|x| > k$
then $|f(x) - L| < \varepsilon$

③ $\lim_{x\to 0^-}$ $\lim_{x\to 0^+}$ $\Rightarrow$ lim exist $\Leftrightarrow$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

$\lim_{x\to a} f(x) = A > 0$

$\exists\ x \in (a-\delta, a+\delta)/(a)$

$f(x) > 0$

### 1.2 Operation of limit

limit exist and finite

$\lim f(x) = A$ $\qquad \lim g(x) = B$

$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$

$u = g(x)$ $\qquad x \to a$ $\qquad u \to b$

$\lim_{x\to a} f(g(x)) = \lim_{u\to b} f(u)$

$\arccos x = t$ $\qquad x = \cos t$

### 1.3 Squeeze theorem

$\lim_{x\to a} f(x) = f(a)$ $\qquad a \notin dom(f)$
$x \in (a-\delta, a+\delta)/(a)$

if $\forall x \in (a-\delta, a+\delta)/\{a\}$ $f(x) = g(x)$
$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

we have $g(x) \le f(x) \le h(x)$

and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \in \mathbb{R}$

then $\lim_{x\to a} f(x) = L$

### 1.4 Two important limits

$\lim_{x\to 0} \frac{x}{\sin x} = 1$

$\lim_{x\to\infty} \left(1 + \frac{1}{g(x)}\right)^{g(x)} = e$

$\Leftrightarrow \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

$\lim_{x\to 0} e^{\frac{1}{x}}$

## 1.5 Infinitesimal and infinite large

$\lim_{x\to a} f(x) = 0 \iff \lim_{x\to a} \frac{k}{f(x)} = \pm\infty$

$\lim_{x\to a} f(x) = \pm\infty \quad \lim_{x\to a} \frac{1}{f(x)} = 0$

## 1.6 Function limit and infinitesimal

$\lim_{x\to a} f(x) = A \quad x \in (a-\delta, a+\delta)$

$f(x) = A + \alpha$

## 1.7 Property of infinitesimal

$\lim_{x\to\infty} \frac{1}{x} + \frac{1}{x^2} = 0$

$\lim_{x\to\infty} \frac{\sin x + \sin x^2 + \cos x^2}{2}$ $-\arctan x$

$\lim_{x\to a} f(x) = 0 \quad g(x) < |M|$

$\Longrightarrow \lim_{x\to a} f(x)g(x) = 0$

## 1.8 Comparison of infinitesimal

$\lim \frac{\alpha}{\beta} = 0 \quad \lim \frac{\alpha}{\beta} = 1 \quad \alpha \sim \beta$

$\lim \frac{\alpha}{\beta} = \infty$

$\lim \frac{\alpha}{\beta} = C$

## 1.9 Equivalent infinitesimal

$\alpha \sim \alpha' \quad \beta \sim \beta' \quad \lim \frac{\alpha'}{\beta'}$ exist

$x \to 0 \quad \lim \frac{\alpha f(x)}{\beta g(x)} = \lim \frac{\alpha' f(x)}{\beta' g(x)}$

(1) $\sin x \sim x$ (2) $\tan x \sim x$

(3) $\arcsin x \sim x$ (4) $\arctan x \sim x$

(5) $1 - \cos x \sim \frac{1}{2}x^2$ (6) $e^x - 1 \sim x$

(7) $\ln(1+x) \sim x$ (8) $(1+x)^\lambda - 1 \sim \lambda x$ $(\lambda \neq 0)$

## 2. Example

### 2.1 Methods for computing limit

$\lim_{x\to +\infty} a^{\frac{1}{x}} = 1 \ (a > 0)$

$\lim_{x\to -\infty} a^{x} = 0 \ (a > 0)$

$\lim_{x\to +\infty} a^{x} = +\infty \ (a > 1)$

$\lim_{x\to +\infty} a^{x} = 0 \ (0 < a < 1)$

$\lim_{x\to +\infty} \ln x = +\infty$

$\lim_{x\to 0^+} \ln x = -\infty$

$\lim_{x\to \pm\infty} \arctan x = \pm\frac{\pi}{2}$

a) Apply four operation of limits

(1) $\lim_{x\to\infty} \frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}$

$\sqrt{1+x} + \sqrt{1-x} + 2$

(2) $\lim_{x\to\infty} \frac{x+\arctan x}{x-\cos x}$

$\lim_{x\to\infty} \frac{1+\frac{1}{x}\arctan x}{1-\frac{1}{x}\cos x} = 1$

(3) $\lim_{x\to -1} \left(\frac{1}{x+1} - \frac{3}{x^3+1}\right)$

$-1$

b) Apply two important limits

(1) $\lim_{x\to -1^+} \frac{(\pi-\arccos x)^2}{1+x}$

$t = \arccos x$

$\cos t = x$

$t \to \pi^-$

$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$

$\cos^2 t + \sin^2 t = 1$

$\sin(\pi - x) = \sin x$

$= \lim_{t\to\pi^-} \frac{(\pi - t)^2}{1+\cos t} \cdot \frac{1-\cos t}{1-\cos t}$

$= \lim_{t\to\pi^-} \frac{(\pi - t)^2}{\sin^2(\pi - t)} (1-\cos t) = 2$

(2) $\lim_{x\to\infty} \left(\frac{x+2}{x+1}\right)^{2x-1}$

$\lim_{x\to\infty} \left(1+\frac{1}{x+1}\right)^{2(x+1)-3}$

$= e^2 \cdot \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x+1}\right)^3} = e^2$

c) Apply squeeze theorem

$$\lim_{x \to 0} sin(\frac{1}{x})\,(x^\wedge 2 + 2x + 3)$$

$-|f(x)| \le \quad \le |f(x)|$

d) Apply infinitesimal

(1) $\lim_{x \to \infty} \frac{x}{x^2+1}(sin\, x + \; arctan\, x)$

(2) $\lim_{x \to 0} \frac{\sqrt{1+sin^2 x} - 1}{(arctan\, x)^2}$ $\quad x^2$

$(1+x)^\lambda - 1 \sim \lambda x$

$(1+g(x))^{\frac{1}{2}} - 1$

$= \lim_{x \to 0} \frac{\frac{1}{2} \sin^2 x}{x^2}$

$= \frac{1}{2}$

2.2 Comparison of infinitesimal

If $f(x) = \sqrt{x+2} - 2\sqrt{x+1} + \sqrt{x}$, find constant c and k, such that when $x \to \infty$, we have $f(x) \sim \frac{c}{x^k}$

$g(x) = \frac{c}{x^k}$

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$

$\frac{c}{x^k}(1+h(x))$

$x \to \infty \quad h(x) \to 1$

$(\sqrt{x+2} - \sqrt{x+1}) - (\sqrt{x+1} - \sqrt{x})$

$= \frac{1}{(\sqrt{x+2} + \sqrt{x+1})} - \left(\frac{1}{(\sqrt{x+1} + \sqrt{x})}\right)$

$k = \frac{3}{2} \quad c = -\frac{1}{4}$

$= \frac{-(\sqrt{x+2} - \sqrt{x})}{\quad} \to (\quad)$

$\frac{-2}{\sqrt{x^3}(2)(2)}$

$(4)$

## 2.3 Limit of piece-wise function

$$f(x) = \begin{cases} \dfrac{ln(1+x)}{x}, x > 0 \\ \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}, x < 0. \end{cases}$$

## 2.4 limit with parameter function

$x^n$

Let $f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n}$ $(x \geq 0)$, find the expression of function f(x).

$0 \leq x < 1$ $\quad f(x) = 0$

$x = 1$ $\quad f(x) = \frac{1}{2}$

$x > 1$ $\quad f(x) = 1$

## 2.5 Definition of Limit

$$\lim_{x\to 2} \frac{x-5}{x+2} = \frac{-3}{4}$$

$\forall \varepsilon > 0$ and $\quad |x-2| < \delta$

$\left| \frac{x-5}{x+2} + \frac{3}{4} \right| = \frac{7}{4} \left| \frac{x-2}{x+2} \right| < \frac{7}{4} \frac{\delta}{|x+2|}$

$-\delta < x-2 < \delta \implies \underbrace{4-\delta}_{0} < x+2 < 4+\delta \quad 0 < \frac{7}{4}\delta$

$\underline{\delta < 4}$ $\quad \frac{1}{|x+2|} < \frac{1}{|4-\delta|} = \varepsilon$

Let $\delta \leq 3$

$\delta = \min(3, \frac{4}{7}\varepsilon)$

Function (1)

1. Basic Knowledge of Function

1.1 Basic concepts of function

Mapping, domain, co-domain, range, graph, formula,

Six types of elementary function

$x > 0$

$\log_a x$

$a < 1$

$\sin x$ $\cos x$ $a^x$ $a > 1$

$a = 1$

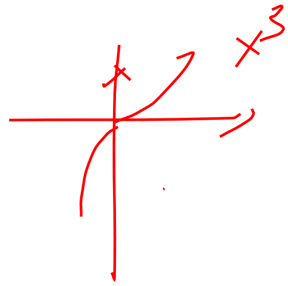
1.2 Operation of functions and the domain of new function

1.3 Monotone; even and odd

$f'(x) > 0$ ↑ $f'(x)$ ↓

$x_1 < x_2$ $f(x_1) < f(x_2)$

$x_1 < x_2$ $f(x_1) > f(x_2)$




$f(-x) = f(x)$ even

$f(-x) = -f(x)$ odd

## 1.4 Continuity of a function (at a point, at an interval)

$\lim_{x \to a} f(x) = f(a)$

$\forall \varepsilon > 0 \quad \exists \delta > 0$

$|x - a| < \delta$

$|f(x) - f(a)| < \varepsilon$

### 1.4.1 Concept of continuity (definition, left/right continuity)

$$\begin{cases} f(a) \text{ defined} \\ \lim_{x \to a} f(x) \text{ exist} \\ \lim_{x \to a} f(x) = f(a) \end{cases}$$

### 1.4.2 Computation of continuous function

### 1.4.3 Properties of continuous function at a closed interval

$[a, b]$

(1) Boundary

(2) Absolute maxima and minima

(3) Intermediate value theorem

$[a, b]$

$c \in (a, b)$

$f(a) < f(c) < f(b)$

(4) Zero point theorem

$[a, b]$

$f(a) f(b) < 0$

$\exists c \in (a, b)$

$f(c) = 0$

$(a, b)$

$$f(x) = \begin{cases} x, & x \in (0, 1] \\ -1 & x = 0 \end{cases}$$

$f(a) f(b) < 0$

2. Example

2.1 Determine continuity of function

$$f(x) = \begin{cases} \dfrac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}, & x \neq 0 \\ 1. & x = 0 \end{cases}$$

2.2 Compute the limit based on the continuity

(1) $\lim\limits_{x \to 0} \frac{ln(1+x)}{x}$

$\ln(1+x) \sim x$

$\lim\limits_{x\to 0} \ln(1+x)^{\frac{1}{x}}$

$\ln[\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}] = 1$

(2) Let function $f(x) = \begin{cases} x^{\alpha} \sin\frac{1}{x}, & x > 0 \\ e^{x} + \beta, & x \leq 0 \end{cases}$ continuous at x=0, find the constants of $\alpha$ and $\beta$.

$\alpha > 0$

$\beta = -1$

$f(0) = 1+\beta$

$\lim\limits_{x\to 0^+} x^{\alpha} \sin\frac{1}{x} = \lim\limits_{x\to 0^-} e^{x} + \beta = 1+\beta$

## 2.3 Properties of continuous function at a closed interval

(1) Show equation $\ln(1+x) = x \cdot 2^{x} - 1$ at least has one real root at the interval (0,1).

$f(x) = g(x)$

$F(x) = f(x) - g(x) = 0$

$[0, 1]$ continuous

$F(0) F(1) < 0$

$F(c) = 0$

(2) Show any real coefficient polynomial function $p(x) = ax^3 + bx^2 + cx + d$ at least has one real root.

$\lim_{x \to -\infty} P(x) = -\infty < 0$ $\quad \exists x_1 \quad P(x_1) < 0$

$\lim_{x \to +\infty} P(x) = +\infty > 0$ $\quad \exists x_2 \quad P(x_2) > 0$

Concept of Derivatives

1. Basic knowledge of concept of derivatives.

1.1 Definition of derivatives

As a limit and as a function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

1.2 geometric interpretation and physical interpretation

Tangent line

$$y - y_0 = f'(x_0)(x - x_0)$$

Normal line

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$$

1.3 Left derivatives and right derivatives

$$f'_-(x) = f'_+(x)$$

1.4 Differentiable and continuity

discontinuous $\begin{cases} f(a) \text{ not define} \\ \lim_{x \to a} f(x) \ \times \\ \lim_{x \to a} f(x) \neq f(a) \end{cases}$

$$\begin{cases} f'_+(x) \neq f'_-(x) \\ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \pm\infty \end{cases}$$

2 Example

$(3x^4)' = 12x^3$

2.1 Find derivatives by definition of derivatives

(1) Let f($\cancel{0}$ x) continuous at x=0, and $\lim_{x\to 0}\frac{f(x)}{x} = 3$, find $f'(0)$.

$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = 3$

$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot \frac{f(x)}{x} = \lim_{x\to 0} x \times \lim_{x\to 0}\frac{f(x)}{x}$

$= 0$

(2) Let $f(x) = \frac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)}$, find $f'(\cancel{0}\ 1)$.

$\frac{(-1)^{n-1}}{n(n+1)}$

$f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$

2.2 Derivatives for piece-wise function

$Let\ f(x) = \begin{cases} e^{x-1}, & x<1 \\ 1+\ln x, & x\geq 1 \end{cases}$, find $f'(1)$.

$f'_-(1) = e^{x-1} = 1$ $\qquad f'(1) = 1$

$f'_+(1) = \frac{1}{x} = 1$

2.2 Derivatives for absolute function

Let $f(x) = 2x^2 + x|x|$, determine whether the function f is differentiable at the point x=0 or not.

$\begin{cases} x^2 & x<0 \\ 3x^2 & x\geq 0 \end{cases}$ $\qquad f'(0) = 0$

$u = \tan x$ $\quad x \to 0$, $u \to 0$

2.4 Find limit based on definition of derivatives.

Let f(x) is differentiable at x=1, and f'(1)=-4. Find $\lim_{x \to 0} \frac{f(1+\tan x) - f(1-2\tan x)}{\sin x}$

continuous

$f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$

$= \lim_{u \to 0} \frac{f(1+u) - f(1-2u)}{u} \quad \lim_{x \to 0} \frac{1}{\cos x}$

$= \lim_{u \to 0} \frac{f(1+u) - f(1)}{u} \quad (-2) \lim_{-2u \to 0} \frac{f(1-2u) - f(1)}{-2u}$

$= f'(1) + 2f'(1)$

$= 12$

2.5 Determine parameter based on the differentiable of a function

Let $f(x) = \begin{cases} e^{ax}, & x \le 0 \\ b(1-x)^2, & x > 0 \end{cases}$ differentiable at x=0, find a, b.

$b = 1. \quad a = 0$

$u \to 0$

$b \le 1$

$-\sin(0) \le f(0) \le \sin(0)$

2.6 Proof by definition of derivatives

$0 \le f(0) \le 0$

Let f(x) differentiable at x=0, and $|f(x)| \le |\sin x|$, prove $|f'(0)| \le 1$.

$|f'(0)| = \left| \lim_{x \to 0} \frac{f(x) - f(0)}{x} \right| = \left| \lim_{x \to 0} \frac{f(x)}{x} \right| \le \lim_{x \to 0} \left| \frac{\sin x}{x} \right|$

$= 1$

2.7 Geometric application of derivatives

Find the equation of the tangent line to the curve with equation $y = e^x$

that through the point (0,0)

$e^0 = 1 \qquad -e^{x_0} = -x_0 e^{x_0}$

$(x_0, y_0) \qquad x_0 = 1$

$y - e^{x_0} = e^{x_0}(x - x_0) \qquad (1, e)$

$\Rightarrow y - e = e(x - 1)$

Computation of Derivatives

1. Basic Knowledge

1.1 Differentiation formulas

(1) Basic formulas

(1) —

$u\,w$

(2) Four fundamental operations

$(u \pm v)' = u'v + v'u$

$(uv)' = u'v + v'u$

$\left(\frac{v}{u}\right)' = \frac{v'u - u'v}{u^2}$

$(Cu)' = Cu'$

(3) Derivative of Inverse function

$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

(4) Derivative of compose function

$[f(g(x))]' = f'(g(x))\,g'(x)$

$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$

$f(g(x)) = y$

$g(x) = u$

(5) Derivative of implicit function

$f(x, y) = 0$

$F(x, y) = z$

$z = 0$

$y' = -\frac{F_x}{F_y}$

$\frac{d\,f(x, y)}{dx} = 0$

$y = y(x)$

$3x^3 + 2x^2y + y^3 = 0$

$0 = 9x^2 + 4xy + 2x^2y' + 3y^2y'$

1.2 Higher derivatives

Definition and Leibniz formula

$f^{(n)}(x)$

$\frac{d^2y}{dx^2}$

$\frac{d^3y}{dx^3}$

$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k\, u^{(n-k)} v^{(k)}$

$v^{(0)} = v$

$u^{(0)} = u$

2 Example (By excises 6. )

2.1 Use differentiation formulas

2.2 Derivatives of piece-wise function

2.3 Derivatives of absolute function

2.4 Derivatives of implicit function

2.6 Derivatives with log function

$f^{(4)} = \sin x + \cos x$

$f^{(60)}(x) = \sin x - \cos x$

2.7 Higher derivatives

$f(x) = \sin(x) + \cos(x)$

$f'(x) = \cos x - \sin(x)$

$f''(x) = -\sin x - \cos x$

$f'''(x) = -\cos x + \sin x$

# Mean Value Theorem

1. Basic Knowledge

1.1 Rolle's Theorem

① $f(x)$ $[a,b]$ con
② $(a,b)$ dif

$f(a) = f(b)$

$\exists\ c,\ f'(c) = 0$

$\in (a,b)$

1.2 Lagrange Mean Value Theorem

① + ② $\Rightarrow$ $\exists\ c \in (a,b)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

1.3 Cauchy's Mean Value Theorem

$f(x)$, $g(x)$ s.t. ① + ② and $g'(x) \neq 0$

$\exists\ c$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

2. Example

2.1 Zero point of derivatives

1) Show that equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ at least has one positive real root smaller than 1.

$c \in (0,1)$ $f'(c) = 0$

$f(x) = ax^4 + bx^3 + cx^2 - (a+b+c)x = 0$

$f(0) = f(1) = 0$

2) Exactly one real root.

at least

at most

$\exists\ (x_1),\ f(x_2) < 0$

$g(x_1) = g(x_2)$

$f(c) = 0$

$f'(c) = 0$

$\Rightarrow >1$ root

contradiction

$\Rightarrow f'(c) \neq 0 \Rightarrow$ 矛盾

### 2.2 Equation with intermediate value

Let f(x) is continuous at [1,2], and it is differentiable at (1,2), and f(1)=1/2, f(2)=2. Show that there exists $c \in (1,2)$, such that $f'(c) = \frac{2f(c)}{c}$.

$\left[\frac{f(c)}{c^2}\right]'$

$f'(c) - \frac{2f(c)}{c} = 0 \Longleftrightarrow 0 = \frac{c^2 f'(c) - 2cf(c)}{c^4}$

$F(x) = \frac{f(c)}{c^2}$ $f(1)$ $f(2)$

$\frac{v'u - uv'}{u^2}$

### 2.3 Equation with end points and intermediate value

Let f(x) is an odd function that differentiable everywhere, show that for any b>0, there exist $c \in (-b, b)$, such that $f'(c) = \frac{f(b)}{b}$.

$f(-x) = -f(x)$

$f(b) - f(-b) = f'(c)(b - (-b))$

$f(b) + f(b) = 2b f'(c)$

### 2.4 Inequality

$f''(x)<0$ $f'(c)=0$ $f(a)$ $f(a+b)$ $f(b)$

For function f(x), we have ~~$f'(x)<0$~~ at [0,c] and f(0)=0. Show that for any constants a,b satisfy 0<a<b<a+b<c, we have f(a)+f(b)>f(a+b)

3 end point

$f'(k_1) = \frac{f(a)-0}{a-0}$

$f'(k_2) = \frac{f(a+b)-f(b)}{a}$

By $f''(x)<0$, f(x) con [0,c] $k_1 \in [0,a]$

f(x) dif (0,c) $k_2 \in [b,a+b]$

$f'(x) \searrow$ on [0,c]

$a(f'(k_1) - f'(k_2)) = f(a)+f(b)-f(a+b)$

$f'(k_1) > f'(k_2)$ ① f(x) [a,b]

$k_1 < k_2$ ↗ ② MVT

$f(b) - f(a) = f'(c)(b-a)$

③ $f'(c)$ ↗ ↘ +

2.5 Equation with two intermediate values

Let f(x) is continuous at [0,1], and it is differentiable at (0,1), and f(0)=0, f(1)=1.

a) Show that there exists $c \in (0,1)$, such that $f(c) = 1 - c$

b) Show that there exist $k, l \in (0,1)$, such that f'(k)f'(l)=1 .

a) $f(c) - 1 + c = 0$

Let $F(x) = f(x) - 1 + x$

By f(x) continuous on [0,1], we have

① F(x) con [0,1]

② $F(1) = 1$ $\quad F(0) = -1$

$F(1) \cdot F(0) < 0$

$\Rightarrow \exists c.$ $f(c) - 1 + c = 0$

b) $0 < k < c < l < 1$

M.V.T

$f'(k) f'(l) = 1$

$k \in [0, c]$

$l \in [c, 1]$

$f'(k) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c)}{c} = \frac{1-c}{c}$

$f'(l) = \frac{f(1) - f(c)}{1 - c} = \frac{1 - f(c)}{1 - c} = \frac{c}{1-c}$

Function (2)

1. Basic Knowledge

1.1 Monotone

$f'(x) < 0$ ↘ on $(a, b)$

$f'(x) > 0$ ↗ on $(a, b)$

1.2 Local Maxima and Local Minima

Fermat Theorem $f'(c) = 0$. $f'(c)$ not exist

Critical number

① $f'(c) = 0$

$f'(x)$ ↗ Local Maxima ↘

$f''(x) > 0$ min

$< 0$ max

$= 0$ indetermine

$f'(d)$ not exist

Local minima

(f

1.3 Absolute Maxima and Absolute Minima

① closed interval method

$f(x)$ con $[a, b]$.

$f(x)$ dif $(a, b)$

① a) critical number c

b) $f(c)$ $f(a)$ $f(b)$

3) biggest Max

smallest Min

② $x \in (-\infty, +\infty)$ / $(a, b)$

$[a, b]$

If c is only critical number that $f'(c) = 0$.

then c local max ⇒ c absolute max

c local min ⇒ c absolute min

2. Example

$\frac{f(x)}{x}$ con $[0,a]$ dif $(0,a)$ MVT

2.1 Let f(x) second differentiable on [0,a], f(0)=0, f''(x)>0. Show that when $0 < x \le a$, function $\frac{f(x)}{x}$ is increasing.

$\left(\frac{f(x)}{x}\right)' = \frac{f'(x)\cdot x - f(x)}{x^2} \Rightarrow f'(x)x - f(x) > 0$

$f'(x) > \frac{f(x)-f(0)}{x-0}$

$\exists c \in (0,x)$ $f'(c) = \frac{f(x)-f(0)}{x-0}$

$\left.\begin{array}{l} x > c \\ f'(x) \uparrow \end{array}\right\} \Rightarrow f'(x) > f'(c)$

2.2 Show that when $x \ge 0$, $\ln(1+x) \ge \frac{\arctan x}{1+x}$ .

$x \in [0, +\infty)$ $x \ge 0$ $\Rightarrow F(x) \ge F(0)$

$F(x) = (1+x)\ln(1+x) - \arctan x \ge 0 = F(0)$

$F'(x) = \frac{1+x}{1+x} + \ln(1+x) - \frac{1}{1+x^2} = \underbrace{\ln(1+x)}_{\ge 0} + \underbrace{\frac{x^2}{1+x^2}}_{\ge 0} \ge 0$

2.3 Let p,q be the constants bigger than 1, and $\frac{1}{p} + \frac{1}{q} = 1$, show that for any x>0, we have $\frac{1}{p}x^p + \frac{1}{q} \ge x.$

$\frac{1}{p}x^p + \left(1 - \frac{1}{p}\right) \ge x$ $x=1$

$\frac{1}{p}x^p - x \ge \frac{1}{p} - 1$

when $x>0$, $f(x) \ge f(1)$ $f(1)$ absolute min

differentiable $x \in (0, +\infty)$

$f'(x) = \frac{1}{p}\cdot p x^{p-1} - 1$

$x=1$ $f'(1) = 0$ $x>1$ $>0$ $\uparrow$

$0<x<1$ $f'(x) < 0$ $\downarrow$ Local min

absolute min