Limit

1. Basic Knowledge of Limit

1.1 the concept of limit

① $\lim_{x\to a} f(x) = L > 0$

② $\lim_{x\to\pm\infty} f(x) = L$

$\lim_{x\to a} f(x) = \pm\infty$

$\lim_{x\to\pm\infty} f(x) = \pm\infty$

① $\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t if $|x-a| < \delta$. then $|f(x)-L| < \varepsilon$

② $\Longleftrightarrow$ $\forall \varepsilon > 0$. $\exists N > 0$, s.t if $|x| > N$, then $|f(x)-L| < \varepsilon$

$\lim_{x\to a^+} f(x) = L$, $\lim_{x\to a^-} f(x) = L$ $\Longrightarrow$ $\lim_{x\to a} f(x)$ exists $\Longleftrightarrow$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \in \mathbb{R}$

$\Longleftarrow$ limit exists and finite

$\lim_{x\to a} f(x) = +\infty$ $\Longleftrightarrow$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = +\infty$

$x \in (a-\delta, a+\delta)\setminus\{a\}$, $f(x) > 0$

1.2 Operation of limit

① $\lim_{x\to a} f(x) = A$ $\quad \lim_{x\to a} g(x) = B$

$\Rightarrow \lim_{x\to a} (f(x) \pm g(x)) = A \pm B$ $\quad \lim_{x\to a} (f(x)\cdot g(x)) = A\cdot B$. $\quad \lim_{x\to a} f(x)/g(x) = A/B$

② $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$ $\quad \Rightarrow g(x) = u$, $u \to b$

$= \lim_{u\to b} f(u)$

if $f(x) = g(x)$ $x \in (a-\delta, a+\delta)/\{a\}$

then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

1.3 Squeeze theorem

$\lim_{x\to a} f(x) = f(a)$ $\quad$ if $a \in dom(f)$

If $\forall x \in (a-\delta, a+\delta)/\{a\}$

we have $g(x) \le f(x) \le h(x)$

and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$

then $\lim_{x\to a} f(x) = L$

1.4 Two important limits

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to\infty} (1+\frac{1}{x})^x = e$ / $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

1.5 Infinitesimal and infinite large

$$\lim_{x \to a} f(x) = 0 \iff \lim_{x \to a} \frac{1}{f(x)} = \pm\infty$$

$$\lim_{x \to a} f(x) = \pm\infty \iff \lim_{x \to a} \frac{1}{f(x)} = 0$$

1.6 Function limit and infinitesimal

$$\lim_{x \to a} f(x) = A \iff f(x) = A + \alpha$$

$x \in (a-\delta, a+\delta) \,/\, \{a\}$

α is infinitesimal

1.7 Property of infinitesimal

sum, product $\Rightarrow$ finite number of infinitesimal is infinitesimal

If $\lim_{x \to a} f(x) = 0$ and $g(x) \leq |M|$, $M \in R^+$,

then $\lim_{x \to a} f(x) g(x) = 0$

1.8 Comparison of infinitesimal

$$\lim \frac{\alpha}{\beta} = 0$$

$$\lim \frac{\alpha}{\beta} = +\infty$$

$$\lim \frac{\alpha}{\beta} = C$$

$$\lim \frac{\alpha}{Cx^r} = 1$$

r ? C ?

$$\lim \frac{\alpha}{\beta} = 1 \iff \alpha \sim \beta$$

1.9 Equivalent infinitesimal

If $\alpha \sim \alpha'$, $\beta \sim \beta'$ $\Rightarrow$ $\lim \frac{\alpha}{\beta} = \lim \frac{\alpha'}{\beta'}$

$x \to 0$.

(1) $\sin x \sim x$ (2) $\arcsin x \sim x$

(3) $\tan x \sim x$ (4) $\arctan x \sim x$

(5) $1 - \cos x \sim \frac{1}{2} x^2$ (6) $e^x - 1 \sim x$

(7) $\ln(1+x) \sim x$ (8) $(1+x)^\lambda - 1 \sim \lambda x$ $(\lambda \neq 0)$

2. Example

2.1 Methods for computing limit

① limit exists
② simplify

$\lim_{x\to+\infty} a^{\frac{1}{x}}=1 \quad (a>0)$

a) Apply four operation of limits

$\lim_{x\to-\infty} a^x = 0 \quad (a>1)$

$\lim_{x\to+\infty} a^x = +\infty \quad (a>1)$

(1) $\lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}$ $\frac{0}{0}$

$\lim_{x\to+\infty} \ln x = +\infty$

$\lim_{x\to 0^+} \ln x = -\infty$

$\lim_{x\to+\infty} \arctan x = \pm\frac{\pi}{2}$

(2) $\lim_{x\to\infty}\frac{x+\arctan x}{x-\cos x}$ $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{1+\frac{\arctan x}{x}}{1-\frac{\cos x}{x}} = 1$

$\lim_{n\to\infty} \frac{(1+a)(1+a^2)\cdots(1+a^{2^n})}{}$

(3) $\lim_{x\to-1}(\frac{1}{x+1}-\frac{3}{x^3+1})$

$= \lim_{x\to 1} (\frac{}{})$

$(1-a)(1+a)=1-a^2$

$(1-\sqrt{f(x)})(1+\sqrt{f(x)})$

$\sin x = \sin(\pi - x)$

b) Apply two important limits

$\arccos x = t \qquad x\to -1^+$

$x = \cos t \quad \Rightarrow t \to \pi^-$

(1) $\lim_{x\to-1^+}\frac{(\pi-\arccos x)^2}{1+x}$

$\lim_{t\to\pi^-}\frac{(\pi-t)^2}{1+\cos t} = \lim_{t\to\pi^-}\frac{(\pi-t)^2\cdot(1+\cos t)}{1-\cos^2 t}$

$\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=1$

$\cos^2\theta+\sin^2\theta=1$

$= \lim_{t\to\pi^-}\frac{(\pi-t)^2}{\sin^2 t}(1-\cos t) = \lim_{t\to\pi^-}\frac{(\pi-t)^2}{\sin^2(\pi-t)}(1-\cos t)$

$= 2$

(2) $\lim_{x\to\infty}(\frac{x+2}{x+1})^{2x-1}$

$\lim_{x\to\infty}(1+\frac{1}{x})^x = e$

$\lim_{x\to\infty}(1+\frac{1}{x+1})^{2(x+1)-3}$

$\lim_{x\to\infty}\left[(1+\frac{1}{x+1})^{x+1}\right]^2 \cdot \frac{1}{(1+\frac{1}{x+1})^3}$

c) Apply squeeze theorem

$f(x)$

$$\lim_{x\to 0} sin(\frac{1}{x})\,(x\text{^}2+2x+3)$$

$-f(x) \le \qquad \le f(x)$

d) Apply infinitesimal

$\times$

(1) $\lim_{x\to\infty}\frac{x}{x^2+1}(sin\ x + arctan\ x)$ $= 0$

$$\frac{\frac{1}{x}}{1+\frac{1}{x}}$$

(2) $\lim_{x\to 0}\frac{\sqrt{1+sin^2x}-1}{(arctan\ x)^2}$

$(1+t^2)^{\frac{1}{2}} - 1$

$(1+x)^{\lambda} - 1 \sim \lambda x$

$arctan\ x \sim x$

$$\lim_{x\to 0}\frac{\frac{1}{2}\sin^2 x}{x^2} = \frac{1}{2}$$

2.2 Comparison of infinitesimal

If $f(x) = \sqrt{x+2} - 2\sqrt{x+1} + \sqrt{x}$, find constant c and k, such that when $x \to \infty$, we have $f(x) \sim \frac{c}{x^k}$

$g(x) = \frac{c}{x^k}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1 \qquad \frac{1}{x^k}$$

$$\lim_{x\to\infty}\frac{1}{x^k c}\left(h(x)\right)$$

$(\sqrt{x+2} - \sqrt{x+1}) - (\sqrt{x+1} - \sqrt{x})$

$h(x) = \frac{-2}{\sqrt{x^3}\,(\ 2 \times 2 \cdot 2\ )}$

$k = \frac{3}{2} \quad c = -\frac{1}{4}$

2.3 Limit of piece-wise function

$\lim_{x\to 0^+} = \lim_{x\to 0^-}$

$$f(x) = \begin{cases} \dfrac{ln(1+x)}{x}, x > 0 \\ \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}, x < 0. \end{cases}$$

$\lim_{x\to 0} f(x)$

2.4 limit with parameter function

$\frac{3^n}{1+3^n}$

Let $f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n}$ $(x \geq 0)$, find the expression of function f(x).

$0 \leq x < 1$ $f(x) = 0$

$x = 1$ $f(x) = \frac{1}{2}$

$x > 1$ $\lim_{n\to\infty} \frac{1}{\frac{1}{x^n}+1} = 1$

2.5 Definition of Limit

$$\lim_{x\to 2} \frac{x-5}{x+2} = \frac{-3}{4}$$

$\forall \varepsilon > 0$ $|x-2| < \delta$

$|f(x) - L| = \left|\frac{x-5}{x+2} + \frac{3}{4}\right| = \frac{7|x-2|}{4|x+2|} < \frac{7\delta}{4}$

$|x+2|$ $-\delta < x-2 < \delta$

$0 < 4-\delta < x+2 < 4+\delta$ $\delta \leq 3$

$\delta < 4$ $\left|\frac{1}{x+2}\right| < \left|\frac{1}{4-\delta}\right|$ $\left|\frac{1}{x+2}\right| < 1$

Function (1)

1. Basic Knowledge of Function

1.1 Basic concepts of function

Mapping, domain, co-domain, range, graph, formula,

Six types of elementary function

$c_n x^n + c_{n-1} x^{n-1} - - c_1 x + c_0$

$\frac{p(x)}{q(x)}$ x^r $r \in R$

$\sin x$ e^x $\log_a x$

1.2 Operation of functions and the domain of new function

$\pm$ $\times$ $/$

$f(g(x))$ $x \in dom(g)$ $\cap$ $g(x) \in dom(f)$ $\Rightarrow x \in [a,b]$

1.3 Monotone; even and odd

$f'(x) \geq 0$ $\uparrow$ $f'(x) < 0$ $\downarrow$

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$f(-x) = f(x)$ $f(-x) = -f(x)$

$\lim_{x \to a} f(x) = f(a)$ $\quad \forall \varepsilon \ \exists \delta > 0 \quad |x - a| < \delta$

$\Rightarrow |f(x) - f(a)| < \varepsilon$

1.4 Continuity of a function (at a point, at an interval)

1.4.1 Concept of continuity (definition, left/right continuity)

$\begin{cases} f(a) \text{ defined} \\ \lim_{x \to a} f(x) \text{ exist} \\ \lim_{x \to a} f(x) = f(a) \end{cases}$

$\lim_{x \to a^-} f(x) = f(a)$

$\lim_{x \to a^+} f(x) = f(a)$

1.4.2 Computation of continuous function

1.4.3 Properties of continuous function at a closed interval

(1) Boundary

$[a, b]$

(2) Absolute maxima and minima

① $\exists$

② closed interval method

(3) Intermediate value theorem

$[a, b]$ $\quad \exists c \in (a, b)$

$f(a) < f(c) < f(b)$

(4) Zero point theorem

$f(a) \times f(b) < 0 \quad f(c) = 0 \quad (0, 1)$

$f(x) = \begin{cases} x & x \in (0, 1] \\ -1 & x = 0 \end{cases}$

2. Example

2.1 Determine continuity of function

$$f(x) = \begin{cases} \dfrac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}, & x \neq 0 \\ 1. & x = 0 \end{cases}$$

$f(0) = 1$

2.2 Compute the limit based on the continuity

(1) $\lim\limits_{x \to 0} \frac{ln(1+x)}{x}$

(2) Let function $f(x) = \begin{cases} x^{\alpha} sin\ \frac{1}{x}, & x > 0 \\ e^{x} + \beta, & x \leq 0 \end{cases}$ continuous at x=0, find the constants of α and β.

$f(0) = 1+\beta$

$\lim\limits_{x \to 0^-} = \lim\limits_{x \to 0^+}$ $\alpha > 0$ $\beta = 1$

2.3 Properties of continuous function at a closed interval

(1) Show equation $\ln(1+x) = x\cdot 2^x - 1$ at least has one real root at the interval (0,1).

$f(x) = g(x)$

$F(x) = f(x) - g(x)$

$F(0)F(1) < 0$

(2) Show any real coefficient polynomial function $p(x) = ax^3 + bx^2 + cx + d$ at least has one real root.

$\lim_{x \to -\infty} P(x) = -\infty$

$\exists x_1$ $P(x_1) < 0$

$\lim_{x \to +\infty} P(x) = +\infty$

$\exists x_2$ $P(x_2) > 0$

Concept of Derivatives

1. Basic knowledge of concept of derivatives.

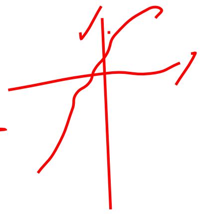
$f'(x)$

1.1 Definition of derivatives

$f'(x_0) =$

As a limit and as a function

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}$$



1.2 geometric interpretation and physical interpretation

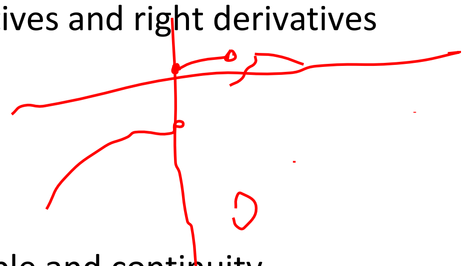
Tangent line

$$y - y_0 = f'(x_0)(x - x_0) \qquad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Normal line

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$

1.3 Left derivatives and right derivatives



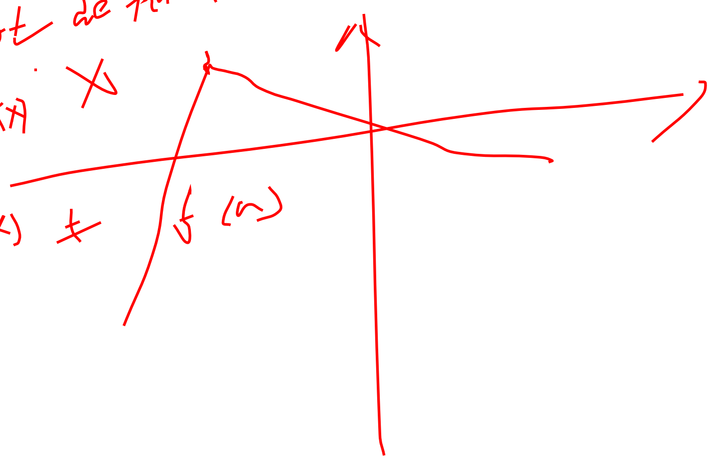
$$\lim_{x \to a^+} f(x) = f(a)$$

1.4 Differentiable and continuity

$\Rightarrow$

discontinuous

- $f(a)$ not defined
- $\lim_{x \to a} f(x)$ X
- $\lim_{x \to a} f(x) \neq f(a)$



$f'_-(a) \neq f'_+(b)$

$f'(a) = \pm\infty$

2 Example

2.1 Find derivatives by definition of derivatives

(1) Let f(0) continuous at x=0, and $\lim\limits_{x\to 0}\frac{f(x)}{x}=3$, find $f'(0)$.

$f(0)=0$

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=3$$

$$\lim_{x\to 0} x\cdot\frac{f(x)}{x}=f(0)$$

(2) Let $f(x)=\frac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)}$, find $\cancel{f'(0)}$. $f'(1)$

$$\lim_{x\to 0} x \cdot \lim_{x\to 0}\frac{f(x)}{x}$$

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$

2.2 Derivatives for piece-wise function

$Let\ f(x)=\begin{cases} e^{x-1}, & x<1 \\ 1+ln\,x, & x\geq 1\end{cases}$, find $f'(1)$.

$x=1$

$$f'_-(1)=e^{x-1}=1$$

$$f'_+(1)=\frac{1}{x}=1$$

2.2 Derivatives for absolute function

Let $f(x)=2x^2+x|x|$, determine whether the function f is differentiable at the point x=0 or not.

$$\begin{cases} 2x^2-x^2 & x<0 \\ 2x^2+x^3 & x>0\end{cases}$$

$x=0$

$f'(0)=0$

$x \to b \quad u \to 0$

$f'(1) = \lim_{\substack{x\to 0 \\ h \to 0}} \frac{f(1+x) - f(1)}{x}$

2.4 Find limit based on definition of derivatives.

$u = \tan x$

Let f(x) is differentiable at x=1, and f'(1)=-4. Find $\lim_{x\to 0}\frac{f(1+\tan x)-f(1-2\tan x)}{\sin x}$.

$\tan x \qquad \frac{\sin}{\cos}$

$\lim_{u\to 0} \frac{f(1+u) - f(1-2u)}{u} \cdot \lim_{x\to 0}\frac{1}{\cos x}$

$= \lim_{u\to 0} \frac{f(1+u) - f(1)}{u} - (-2)\lim_{-2u\to 0}\frac{f(1-2u) - f(1)}{-2u} = f'(1) + 2f'(1) = 3f'(1)$

2.5 Determine parameter based on the differentiable of a function

Let $f(x) = \begin{cases} e^{ax}, & x \le 0 \\ b(1-x)^2, & x > 0 \end{cases}$ differentiable at x=0, find a, b.

$(e^{ax})' = (b(1-x)^2)' \quad x = 0$

$a = -2$

$b = 1$

2.6 Proof by definition of derivatives

$a > 0$

$b \le 1$

$|f(0)| \le |\sin 0|$

Let f(x) differentiable at x=0, and $|f(x)| \le |\sin x|$, prove $|f'(0)| \le 1$.

$|f'(0)| = \left|\lim_{x\to 0}\frac{f(x) - f(0)}{x}\right| = \left|\lim_{x\to 0}\frac{f(x)}{x}\right| \qquad 0 \le |f(0)| \le 0$

$\le \lim_{x\to 0}\left|\frac{\sin x}{x}\right| \qquad f(0) = 0$

2.7 Geometric application of derivatives

$= 1$

Find the equation of the tangent line to the curve with equation $y = e^x$ that through the point (0,0)

$e^0 = 1 \qquad (0,0)\ X$

Let tangent point is (x_0, y_0)

$x_0 = 1$

$\Rightarrow y_0 = e$

we have — $y - y_0 = e^{x_0}(x - x_0)$

by (0,0) $\quad -e^{x_0} = e^{x_0}(-x_0)$

$y - e = e(x-1) \qquad x_0 = 1$

Computation of Derivatives

1. Basic Knowledge

1.1 Differentiation formulas

(1) Basic formulas

(1) (2)

(15) — (16)

(2) Four fundamental operations

$(u \pm v)' = u' \pm v'$

$(u \cdot v)' = u'v + v'u$

$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

$(cu)' = cu'$

(3) Derivative of Inverse function

$y = f^{-1}(x)$

$$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

(4) Derivative of compose function

$y = f(g(x))$

$$[f(g(x))]' = f'(g(x))\, g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$u = g(x)$

$y = y(x)$

(5) Derivative of implicit function

$f(x, y) = 0$

$F(x, y) - z$

$z = 0$

$$\begin{cases} \frac{dy}{dx} = -\frac{F_x}{F_y} \\ \frac{d f(x, y)}{dx} = 0 \end{cases}$$

$$x^3 + 2xy + y^2 = 0$$

$$3x^2 + 2y + 2xy' + 2yy' = 0$$

(48)

1.2 Higher derivatives

Definition and Leibniz formula

$f^{(n)}(x)$

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} v^{(k)}$$

$\frac{d^2 y}{dx^2}$ $\frac{d^3 y}{dx^3}$

$y = \sin x + \cos x$

$y' = \cos x - \sin x$

$y^{(2)} = -\sin x - \cos x$

$y^{(3)} = -\cos x + \sin x$

$y^{(4)} = y$

$y^{(50)} = -\sin x - \cos x$

2 Example (By excises 6.)

2.1 Use differentiation formulas

2.2 Derivatives of piece-wise function

2.3 Derivatives of absolute function

2.4 Derivatives of implicit function

2.6 Derivatives with log function

2.7 Higher derivatives

Mean Value Theorem

1. Basic Knowledge

1.1 Rolle's Theorem

① $f(x)$ $[a,b]$ continuous

② (a,b) diff

③ $f(a)=f(b)$

$\Rightarrow \exists c \in (a,b)$ $f'(c)=0$

1.2 Lagrange Mean Value Theorem

① + ②

$\Rightarrow \exists c \in (a,b)$ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

1.3 Cauchy's Mean Value Theorem

$f(x)$ $g(x)$ ① + ②

$\Rightarrow \exists c \in (a,b)$ $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

2. Example

2.1 Zero point of derivatives

1) Show that equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ at least has one positive real root smaller than 1.

$\exists k \in (0,1)$ ~~$f(k)=0$~~ $f'(k)=0$

$f(x) = ax^4 + bx^3 + cx^2 - (a+b+c)x$. $f(0)=f(1)=0$

2) Exactly one real root.

at least $\begin{cases} f(a)f(b)<0 \text{ and } [a,b] \text{ continuous} \\ g(a)=g(b) \text{ and ① ②} \end{cases}$ $\Rightarrow f(c)=0$, $g'(c)=0$

at most Contradiction $\exists x_2$ $f(c)=0$, $f(x_2)=0$

$k \in (c, x_2)$ $f'(k) \neq 0$ $\Rightarrow$ 矛盾

$f(x_1) = 0$ suppose x_2, $f(x_2) = 0$ $f'(x) \neq 0$

$f(x_1) = f(x_2) = 0 \Rightarrow \exists k \in (x_1, x_2)$

$f'(k) = 0$ 矛盾

2.2 Equation with intermediate value

$v = f(x)$ $u = x^2$

Let f(x) is continuous at [1,2], and it is differentiable at (1,2), and f(1)=1/2, f(2)=2. Show that there exists $c \in (1,2)$, such that $f'(c) = \frac{2f(c)}{c}$.

$\frac{x^2 f'(x) - 2x f(x)}{(x^2)^2} = 0$

$f'(x) - \frac{2f(x)}{x} = 0 \Longleftrightarrow$

$F(x) = \frac{f(x)}{x^2}$

$F(x)$ ① ②

$F(1) = F(2) = \frac{1}{2}$

$\left(\frac{v}{u}\right)' = \frac{v'u - u'v}{u^2}$

2.3 Equation with end points and ~~intermediate value~~

Let f(x) is an odd function that differentiable everywhere, show that for any b>0, there exist $c \in (-b, b)$, such that $f'(c) = \frac{f(b)}{b}$.

$f(-x) = -f(x)$

MVT

$\exists k$ $f'(k) = \frac{f(b) - f(-b)}{2b}$

$= \frac{f(b) + f(b)}{2b} = \frac{f(b)}{b}$

2.4 Inequality

$f''(x) < 0$ con dif

For function f(x), we have $f'(x)<0$ at [0,c] and f(0)=0. Show that for any constants a,b satisfy 0<a<b<a+b<c, we have f(a)+f(b)>f(a+b)

By $f''(x) < 0$ at $[0, c]$

$\Rightarrow$ $f(x)$ con — $[0, c]$; dif — $(0, c)$

$k_1 \in [0, a]$ $k_2 \in [b, a+b]$

$f'(k_1) = \frac{f(a) - f(0)}{a} < 0$

$f'(k_2) = \frac{f(a+b) - f(b)}{a} < 0$

$\frac{f(b) - f(b) = f'(c)(b-a)}{a(f'(k_2) - f'(k_1)) =}$ $\frac{f(a+b) - f(b) - f(a)}{} < 0$

$f'(k_2) - f'(k_1) < 0$

$f'(k_2) < f'(k_1)$

$f''(x) < 0$ and $k_1 < k_2 \Rightarrow$

2.5 Equation with two intermediate values

Let f(x) is continuous at [0,1], and it is differentiable at (0,1), and f(0)=0, f(1)=1.

a) Show that there exists $c \in (0,1)$, such that $f(c) = 1 - c$

$c \in (a.b)$

$\cancel{f'(c)}$ $f(c)$

b) Show that there exist $k, l \in (0,1)$, such that $f'(k)f'(l)=1$.

a) $f(c) - 1 + c = 0$

$F(x) = f(x) - 1 + x$

① $F(x)$ con $[0.1]$

② $\begin{cases} F(0) = -1 > 0 \\ F(1) = 1 < 0 \end{cases}$

$F(0)F(1) < 0$

$\Rightarrow$ by zero point theorem

b) Let $0 < k < c < l < 1$

$f'(k) \cdot f'(l) = 1$

$f'(k) = \dfrac{f(c) - 0}{c - 0}$

$f'(k) \in [0.c]$

$f'(l) = \dfrac{1 - f(c)}{1 - c}$

$f'(l) \in [c.1]$

$\dfrac{1-c}{\cancel{f(c)}} \Big/ c \cdot \dfrac{c}{\cancel{1-f(c)}} \Big/ (1-c) = 1$

Function (2)

1. Basic Knowledge

1.1 Monotone

$f'(c) > 0$ ↗ $c \in (a,b)$

$f'(c) < 0$ ↘ $c \in (a,b)$

1.2 Local Maxima and Local Minima

① closed interval method

critical number

② critical number ⟹ Local Maxima ↗ ↘ Local Minima ↘ ↗

$f'(x)$ $f(x)$ <0 >0

③ $f(x)$ differentiable at (a,b) only c. $f'(x) = 0$ $f'(c) = 0$. c Local max Local min

1.3 Absolute Maxima and Absolute Minima

① closed interval method c ⟹ absolute max min

② only $f'(c) = 0$ local ⟹ absolute

2. Example

2.1 Let f(x) second differentiable on [0,a], f(0)=0, f''(x)>0. Show that when $0 < x \le a$, function $\frac{f(x)}{x}$ is increasing.

$f'(x) > \frac{f(x)-f(0)}{x-0}$

$f'(x) > f'(c)$

$[0, a)$

$\left(\frac{f(x)}{x}\right)' = \frac{xf'(x) - f(x)}{x^2} > 0 \Rightarrow xf'(x) - f(x) > 0$

$\Uparrow$

$\exists c \in (0, x)$

$xf'(c) = f(x) - f(0) = f(x)$

$f'(c)$

$f'(x) \nearrow$

$xf'(x) > xf'(c) = f(x)$

2.2 Show that when $x \ge 0$, $\ln(1+x) \ge \frac{\arctan x}{1+x}$.

$[0, x)$

$f(x) = (1+x)\ln(1+x) - \arctan x$

$f(x) \ge f(0) = 0$

$f'(x) \ge 0$ $\quad \frac{1+x}{1+x} + \ln(1+x) - \frac{1}{1+x^2} \Rightarrow \underbrace{\ln(1+x)}_{\ge 0} + \underbrace{\frac{x^2}{1+x^2}}_{\ge 0} \ge 0$

2.3 Let p,q be the constants bigger than 1, and $\frac{1}{p} + \frac{1}{q} = 1$, show that for any x>0, we have $\frac{1}{p}x^p + \frac{1}{q} \ge x$.

$x = 1$

$\frac{1}{p}x^p + \left(1 - \frac{1}{p}\right) \ge x$

$\frac{1}{p}x^p - x \ge \frac{1}{p} - 1$

$\forall x > 0$

$f(x) = \frac{1}{p}x^p - x$

$(0, +\infty)$

$f(x) \ge f(1)$

$x < 1$ $\quad x = 1$ $\quad$ ~~f~~ $x > 1$

$f'(x) = p \cdot \frac{1}{p} x^{p-1} - 1$

$f'(x) = 0$

$\searrow$ Local min $\nearrow$

absolute min