高等数学习题课

马文俊

华南师范大学 阿伯丁人工智能与大数据学院

October 17, 2022

1.Suppose that f is an even function. What can you conclude about f'? Prove your answer.

Answer to the 1st question

It follows that f' is an odd function, because

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{-h} \\ &= -\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= -f'(x). \end{aligned}$$

Here the third equation came from substituting h for $-h$ in the limit. (Why is this valid?)

2. Express the following in the form x^c for an appropriate choice of c.

$\sqrt[n]{x}$ $\qquad (x^a)^b$ $\qquad x^a x^b$ $\qquad \dfrac{1}{x^a}$ $\qquad \sqrt[3]{x}\sqrt[4]{x}$ $\qquad \dfrac{1}{x^{5/2}\sqrt[3]{x}}$

Answer to the 2nd question

$x^{1/n}$ $\quad$ x^{ab} $\quad$ x^{a+b} $\quad$ x^{-a} $\quad$ $x^{7/12}$ $\quad$ $x^{-17/6}$

3. a) Find the derivative of $f(x) = (2x^2 - 1)(x^2 + x)$ in two ways: by using the Product Rule and by multiplying first. Do the answers agree? Which method is faster?

b) Find the derivative of $f(x) = \dfrac{2x^3 + \sqrt{x} - x}{x^2}$ in two ways: by using the Quotient Rule and by simplifying first. Do the answers agree? Which method is faster?

Answer to the 3rd question

a) Both methods give $f'(x) = 8x^3 + 6x^2 - 2x - 1$. I found it quicker to multiply first and then differentiate.

b) Both methods give $f'(x) = 2 - \frac{3}{2}x^{-\frac{5}{2}} + x^{-2}$. It is faster to simplify first.

4. Find the equation of the tangent line to the curve at the given point.

$y = \dfrac{x}{2x-1}$, $(1, 1)$.

$y = x^3 + 2x^2 + 1$, $(1, 4)$.

Answer to the 4rd question

Worked Solution: $y = \dfrac{x}{2x-1}$, so

$$\frac{dy}{dx} = \frac{(2x-1)\cdot 1 - x\cdot(2-0)}{(2x-1)^2} = \frac{2x-1-2x}{(2x-1)^2} = \frac{-1}{(2x-1)^2}.$$

So when $x = 1$ we have $\dfrac{dy}{dx} = \dfrac{-1}{(2\cdot 1 - 1)^2} = -1$. Thus the equation of the tangent line through $(1,1)$ is $y - 1 = -1(x-1)$ or in other words $y = -x + 2$.

$y = 7x - 3$

5. Remember the special limits

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1, \qquad \lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0.$$

Use the first one to compute the following.

$$\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} \qquad \lim_{\theta \to 0} \frac{\theta^2}{\sin(\theta)} \qquad \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta + \sin(\theta)}$$

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta + \theta^2}$$

Worked Solution:

$$\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}=\lim_{\theta\to 0}\frac{1}{\frac{\sin(\theta)}{\theta}}=\frac{1}{\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}}=\frac{1}{1}=1.$$

$\lim_{\theta\to 0}\frac{\theta^2}{\sin(\theta)}=0.$

Worked Solution: We divide top and bottom of the fraction by θ so that the expression now consists of terms whose limits we know.

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta+\sin(\theta)}=\lim_{\theta\to 0}\frac{\frac{\sin(\theta)}{\theta}}{1+\frac{\sin(\theta)}{\theta}}=\frac{\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}}{1+\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}}=\frac{1}{1+1}=1/2$$

$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta+\theta^2}=1.$

Thank you!