3-9. Integrating rational functions: partial fraction decomposition

We already know of a procedure to integrate polynomials. Here we shall see how to integrate a "rational function", i.e., a quotient of one polynomial by another.

Occasionally, there might be a substitution available to solve a complicated rational function, e.g.,

$$\int \frac{3x^2 + x}{2x^3 + x^2 - 7}\, dx = \int \frac{1}{u}\,\frac{du}{2} \quad \text{(subst. } u = 2x^3 + x^2 - 7\text{)}$$

$$= \ln|u| + C$$

$$= \ln\left|2x^3 + x^2 - 7\right| + C.$$

Usually, this approach won't work, e.g., with

$$\int \frac{5x - 15}{x^2 + 3x - 4}\, dx.$$

This integral can be solved easily, however, by noticing

$$\frac{7}{x+4} - \frac{2}{x-1} = \frac{7(x-1) - 2(x+4)}{x^2 + 3x - 4} = \frac{5x - 15}{x^2 + 3x - 4} \tag{4}$$

so that

$$\int \frac{5x - 15}{x^2 + 3x - 4}\, dx = \int \frac{7}{x+4} - \frac{2}{x-1}\, dx$$

$$= 7\ln|x+4| - 2\ln|x-1| + C.$$

Finding the decomposition (4) might look, at first, like a stroke of luck.

In fact, there is a systematic way to do this. We first note that the denominator of the integrand, $x^2 + 3x - 4$, factors as

$$x^2 + 3x - 4 = (x + 4)(x - 1).$$

Then we try to solve

$$\frac{5x - 15}{x^2 + 3x - 4} = \frac{A}{x + 4} + \frac{B}{x - 1} \tag{5}$$

for real numbers A and B.

Multiplying the equation by $(x^2 + 3x - 4)$ produces

$$5x - 15 = A(x - 1) + B(x + 4) = (A + B)x + (-A + 4B).$$

We view this as a system of two equations in two unknowns,

$$\begin{aligned} (1):\quad && 5 &= A + B, \\ (2):\quad && -15 &= -A + 4B \end{aligned}$$

Now we solve this system:

$$\begin{aligned} (1)+(2):\quad && -10 &= 5B, \\ && B &= -2, \\ (1):\quad && 5 &= A + (-2), \\ && A &= 7. \end{aligned}$$

Plugging this solution into (5) yields

$$\frac{5x-15}{x^2+3x-4} = \frac{7}{x+4} - \frac{2}{x-1}.$$

This sort of thing can be done very generally, which is the content of the following theorem:

Theorem 3.38 (Partial Fraction Decomposition). *Let $f(x)$ be a rational function,*

$$f(x) = \frac{p(x)}{q_1(x)^{n_1} \cdots q_k(x)^{n_k} r_1^{m_1}(x) \cdots r_l^{m_l}}(x),$$

where $p(x)$ is a polynomial and q_i are distinct monic linear polynomials,

$$q_i(x) = x + a_i, \quad i = 1, \dots, k$$

and the $r_i(x)$ are distinct monic quadratic polynomials,

$$r_i(x) = x^2 + b_i x + c_i \text{ where } b_i^2 - 4c_i < 0, \quad i = 1, \dots, l.$$

Then $f(x)$ decomposes as

$$f(x) = g(x) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{A_{i,j}}{(x + a_i)^j}$$
$$+ \sum_{i=1}^{l} \sum_{j=1}^{m_i} \frac{B_{i,j}x + C_{i,j}}{(x^2 + b_i x + c_i)^j},$$

for some polynomial $g(x)$ and some real numbers $A_{i,j}, B_{i,j}, C_{i,j}$.

We do not prove this theorem in this course. The statement of this theorem is fairly involved, so let us break it down a bit, and explain how to use it.

We start with a general rational function

$$f(x) = \frac{p(x)}{q(x)}.$$

We may assume that $q(x)$ is a monic polynomial (i.e., the coefficient on its first term is 1).

By the Fundamental Theorem of Algebra, $q(x)$ factors into irreducible linear and quadratic terms. (In practice, given a "random" polynomial $q(x)$, it may be extremely difficult to factor it; in some cases, it is not even possible to factor it exactly. However, in all examples in this course, this factorisation will be possible and not too difficult.)

There may be some factors repeated in the factorisation of $q(x)$; these need to be collected together. Altogether, this yields a factorisation

$$q(x) = q_1(x)^{n_1} \cdots q_k(x)^{n_k} r_1(x)^{m_1} \cdots r_l(x)^{m_l}$$

where $q_i(x)$ and $r_i(x)$ are as described in the above theorem.

We can then expect to express $f(x)$ as a sum of the following terms:

1. Some polynomial, $g(x)$.
2. For every linear term $q_i(x) = x + a_i$ in the factorisation of $q(x)$ (occurring n_i times) terms of the form

$$\frac{A_{i,j}}{(x + a_i)^j},$$

where $j = 1, \ldots, n_i$. (If $n_i = 1$, i.e., if $(x + a_i)$ only appears once in the factorisation of $q(x)$, then we just have one term here.)

3. For every quadratic term $r_i(x) = x^2 + b_i x + c_i$ in the factorisation of $q(x)$ (occurring m_i times) terms of the form

$$\frac{B_{i,j}x + C_{i,j}}{(x^2 + b_i^x + c_i)^j}$$

where $j = 1, \ldots, m_i$. (Again, if $m_i = 1$, i.e., if $(x^2 + b_i x + c_i)$ only appears once in the factorisation of $q(x)$, then we just have one term here.)

Let us do some examples of how to express rational functions using this theorem. After this, we will go on to integrating rational functions.

Example 3.39. *Find the partial fraction decomposition of the following rational functions:*

(i) $$\frac{x+7}{x^2+5x+6}.$$

(ii) $$\frac{x+7}{x^2+4x+4}.$$

(iii) $$\frac{31x+51}{x^3+4x^2-15x-18}.$$

(iv) $$\frac{x^4+x^3+5x^2-10x-1}{(x+3)(x-1)^2}.$$

(v) $$\frac{x^3+7x^2+6x+4}{x^3-1}.$$

Solution. (i): First factor the denominator:

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

Hence we want to solve

$$\frac{x + 7}{x^2 + 5x + 6} = g(x) + \frac{A}{x + 2} + \frac{B}{x + 3}$$

for a polynomial $g(x)$ and scalars $A, B \in \mathbb{R}$.
Multiplying both sides by $(x^2 + 5x + 6)$ produces

$$\begin{aligned} x + 7 &= g(x)(x^2 + 5x + 6) + A(x + 3) + B(x + 2) \\ &= g(x)(x^2 + 5x + 6) + (A + B)x + (3A + 2B). \end{aligned}$$

If the polynomial $g(x)$ were nonzero, then the right-hand side would have some term involving x^k for some $k \geq 2$. Since the left-hand side has degree one, the polynomial $g(x)$ must be zero.

This gives the system of equations

$$\begin{aligned}(1): \quad 1 &= A + B,\\ (2): \quad 7 &= 3A + 2B.\end{aligned}$$

We solve this as follows:

$$\begin{aligned}(2) - 3(1): \quad 4 &= -B\\ B &= -4\\ (1): \quad 1 &= A - 4,\\ A &= 5.\end{aligned}$$

We therefore have

$$\frac{x+7}{x^2+5x+6} = \frac{5}{x+2} - \frac{4}{x+3}.$$

(ii): The denominator factors as

$$x^2 + 4x + 4 = (x + 2)^2.$$

Hence we want to solve

$$\frac{x+7}{x^2+4x+4} = g(x) + \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

for a polynomial $g(x)$ and scalars $A, B \in \mathbb{R}$.
Multiplying this equation by $(x+2)^2$ produces

$$x + 7 = g(x)(x+2)^2 + A(x+2) + B = g(x)(x+2)^2 + Ax + (2A + B).$$

Again, $g(x)$ must be zero since the left-hand side has degree < 2. This system solves easily:

$$A = 1, \quad B = 7 - 2 = 5.$$

Therefore,

$$\frac{x+7}{x^2+4x+4} = \frac{1}{x+2} + \frac{5}{(x+2)^2}.$$

(iii): First factor the denominator:

$$x^3 + 4x^2 - 15x - 18 = (x-3)(x+1)(x+6)$$

Hence we want to solve

$$\frac{31x+51}{x^3+4x^2-15x-18} = g(x) + \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{x+6}$$

for a polynomial $g(x)$ and scalars $A, B, C \in \mathbb{R}$.
Multiplying this equation by $(x^3 + 4x^2 - 15x - 18)$ produces

$$\begin{aligned} 31x + 51 = g(x)&(x^3 + 4x^2 - 15x - 18) + A(x+1)(x+6) + B(x-3)(x+6) \\ &+ C(x-3)(x+1) \\ = g(x)&(x^3 + 4x^2 - 15x - 18) + (A+B+C)x^2 + (7A + 3B - 2C)x \\ &+ (6A - 18B - 3C). \end{aligned}$$

Since the left-hand side has degree < 3, the polynomial $g(x)$ must be zero.
This gives a system of three equations,

$$\begin{array}{lrcl} (1): & 0 & = & A + B + C, \\ (2): & 31 & = & 7A + 3B - 2C, \\ (3): & 51 & = & 6A - 18B - 3C. \end{array}$$

We now solve this system:

$$\begin{array}{rrcl}
(4):\ (2)+2(1): & 31 & = & 9A+5B, \\
(5):\ (3)+3(1): & 51 & = & 9A-15B, \\
(5)-(4): & 20 & = & -20B, \\
 & B & = & -1, \\
(4): & 31 & = & 9A-5, \\
 & A & = & 36/9=4, \\
(1): & 0 & = & 4-1+C, \\
 & -3 & = & C.
\end{array}$$

Therefore,

$$\frac{31x+51}{x^3+4x^2-15x-18} = \frac{4}{x-3} - \frac{1}{x+1} - \frac{3}{x+6}.$$

(iv): The denominator has already been factored, so we go straight to the partial fraction decomposition with variables to be solved. We want to solve

$$\frac{x^4+x^3+5x^2-10x-1}{(x+3)(x-1)^2} = g(x) + \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Multiplying by $(x+3)(x-1)^2$ yields

$$x^4+x^3+5x^2-10x-1 = g(x)(x+3)(x-1)^2 + A(x-1)^2 + B(x+3)(x-1) + C(x+3).$$

Here, the left-hand side has degree 4, while the component of the right-hand side that doesn't involve $g(x)$ can only have degree at most 2. Hence, we need the polynomial $g(x)$ to have degree 1, so we let $g(x) = Dx + E$, giving

$$\begin{aligned} &x^4+x^3+5x^2-10x-1 \\ &\quad = (Dx+E)(x+3)(x-1)^2 + A(x-1)^2 + B(x+3)(x-1) + C(x+3) \\ &\quad = Dx^4 + (D+E)x^3 + (A+B-5D+E)x^2 \\ &\qquad + (-2A+2B+C+3D-5E)x + (A-3B+3C+3E), \end{aligned}$$

translating into the system

$$\begin{array}{lrcl} (1): & 1 & = & D, \\ (2): & 1 & = & D+E, \\ (3): & 5 & = & A+B-5D+E, \\ (4): & -10 & = & -2A+2B+C+3D-5E, \\ (5): & -1 & = & A-3B+3C+3E. \end{array}$$

We now solve this system:

$$
\begin{array}{rrcl}
(2): & 1 & = & 1+E, \\
 & E & = & 0, \\
(6):\ (5)-3(4): & 29 & = & 7A-9B-9(1), \\
 & 38 & = & 7A-9B, \\
(6)+9(3): & 83 & = & 16A-45 \\
 & A & = & 128/16 = 8, \\
(3): & 5 & = & 8+B-5, \\
 & B & = & 2, \\
(5): & -1 & = & 8-3(2)+3C \\
 & C & = & -3/3 = -1.
\end{array}
$$

Therefore,

$$\frac{x^4+x^3+5x^2-10x-1}{(x+3)(x-1)^2} = x + \frac{8}{x+3} + \frac{2}{x-1} - \frac{1}{(x-1)^2}.$$

Generally, when the degree of the numerator is below the degree of the denominator, the polynomial term ("$g(x)$") will be zero in the Partial Fraction Decomposition. Otherwise, the degree of $g(x)$ is the difference between the degree of the numerator and the degree of the numerator.

(v): The denominator factors as

$$(x-1)(x^2+x+1),$$

and we know that we can't factor x^2+x+1, since the discriminant is

$$1^2-4(1)=-3<0.$$

We want to solve

$$\frac{x^3+7x^2+6x+4}{x^3-1}=g(x)+\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.$$

Multiplying by x^3-1 yields

$$x^3+7x^2+6x+4=g(x)(x^3-1)+A(x^2+x+1)+(Bx+C)(x-1).$$

Since the right-hand side has degree 3, we see that we need $g(x)$ to be a constant, $g(x)=D$. The above equation becomes

$$\begin{aligned} x^3+7x^2+6x+4 &= D(x^3-1)+A(x^2+x+1)+(Bx+C)(x-1) \\ &= Dx^3+(A+B)x^2+(A-B+C)x+(-D+A-C). \end{aligned}$$

Our system is thus

$$\begin{aligned} (1): \quad 1 &= D, \\ (2): \quad 7 &= A+B, \\ (3): \quad 6 &= A-B+C, \\ (4): \quad 4 &= A-C-D \\ &= A-C-1. \end{aligned}$$

We solve this system:

$$
\begin{array}{rrcl}
(2)+(3)+(4): & 17 & = & 3A-1, \\
 & A & = & 18/3 = 6, \\
(2): & 7 & = & 6+B, \\
 & B & = & 1, \\
(3): & 6 & = & 6-1+C, \\
 & C & = & 1.
\end{array}
$$

Therefore,

$$
\frac{x^3+7x^2+6x+4}{x^3-1} = 1 + \frac{6}{x-1} + \frac{x+1}{x^2+x+1}.
$$

Once we've decomposed a rational function using Partial Fraction Decomposition, we should be able to integrate the various terms that appear. Here are some examples.

Example 3.40. *Solve the following integrals.*

(i) $$\int \frac{x+7}{x^2+5x+6}\,dx.$$

(ii) $$\int \frac{x+7}{x^2+4x+4}\,dx.$$

(iii) $$\int \frac{x^4+x^3+5x^2-10x-1}{(x+3)(x-1)^2}\,dx.$$

(iv) $$\int \frac{7x^2-x+4}{x^3+x}\,dx.$$

(v) $$\int \frac{x^3-7x^2}{x^2-6x+10}\,dx.$$

(vi) $$\int \frac{4x^3-8x^2-x+1}{16x^4+8x^2+1}\,dx.$$

Solution. (i): Using Example 3.39 (i), we get

$$\int \frac{x+7}{x^2+5x+6}\,dx = \int \frac{5}{x+2} - \frac{4}{x+3}\,dx$$
$$= 5\ln|x+2| - 4\ln|x+3| + C.$$

(ii): Using Example 3.39 (ii), we get

$$\int \frac{x+7}{x^2+4x+4}\,dx = \int \frac{1}{x+2} + \frac{5}{(x+2)^2}\,dx$$
$$= \ln|x+2| - \frac{5}{x+2} + C.$$

(iii): Using Example 3.39 (ii), we get

$$\int \frac{x^4+x^3+5x^2-10x-1}{(x+3)(x-1)^2}\,dx = \int x + \frac{8}{x+3} + \frac{2}{x-1} - \frac{1}{(x-1)^2}\,dx$$
$$= x^2 + 8\ln|x+3| + 2\ln|x-1| + \frac{1}{x-1} + C.$$

(iv): We need to do Partial Fraction Decomposition on the integrand. The denominator factors as

$$x^3 + x = x(x^2 + 1),$$

so we want to solve

$$\frac{7x^2 - x + 4}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

(We see that we don't need to add a polynomial, because the degree of the numerator is less than the degree of the denominator.)
We get

$$7x^2 - x + 4 = A(x^2 + 1) + (Bx + C)x = (A + B)x^2 + Cx + A,$$

which easily solves as

$$A = 4, \quad B = 3, \quad C = -1.$$

Therefore,

$$\int \frac{7x^2 - x + 4}{x^3 + x}\, dx = \int \frac{4}{x} + \frac{3x - 1}{x^2 + 1}\, dx.$$

to integrate $\frac{3x-1}{x^2+1}$, we separate it into two sums, using the substitution $u = x^2 + 1$ for the first.

$$\int \frac{7x^2 - x + 4}{x^3 + x}\, dx = \int \frac{4}{x} + \frac{3x - 1}{x^2 + 1}\, dx$$

$$= 4 \ln|x| + \frac{3}{2} \ln(x^2 + 1) - \tan^{-1}(x) + C.$$

(v): The denominator in this case cannot be further factored, since the discriminant is

$$(-6)^2 - 4(10) = -4 < 0.$$

So we just need to write the integrand as

$$\begin{aligned}\frac{x^3 - 7x^2}{x^2 - 6x + 10} &= \frac{Ax + B}{x^2 - 6x + 10} + Cx + D,\\ x^3 - 7x^2 &= Ax + B + (Cx + D)(x^2 - 6x + 10)\\ &= Cx^3 + (-6C + D)x^2 + (A + 10C - 6D)x + (B + 10D).\end{aligned}$$

This gives the system

$$\begin{aligned}1 &= C,\\ -7 &= -6C + D,\\ 0 &= A + 10C - 6D,\\ 0 &= B + 10D,\end{aligned}$$

which we can solve,

$$C = 1, \quad , D = -1, \quad A = -16, \quad B = 10.$$

Therefore,

$$\frac{x^3 - 7x^2}{x^2 - 6x + 10} = \frac{-16x + 10}{x^2 - 6x + 10} + x - 1.$$

When we integrate this, we'll want to use the substitution

$$u = x^2 - 6x + 10, \quad du = (2x - 6)\,dx$$

to deal with part of the fractional term. We therefore break apart this fractional term,

$$\frac{-16x + 10}{x^2 - 6x + 10} = -8 \cdot \frac{2x - 6}{x^2 - 6x + 1} - \frac{38}{x^2 - 6x + 10}.$$

The second term here we handle by completing the square in the denominator:

$$x^2 - 6x + 10 = (x - 3)^2 + 1.$$

Putting this together, we get

$$\begin{aligned}
&\int \frac{x^3 - 7x^2}{x^2 - 6x + 10}\,dx \\
&\quad = -8 \int \frac{2x - 3}{x^2 - 6x + 1}\,dx - 38 \int \frac{1}{(x - 3)^2 + 1}\,dx + \int x - 1\,dx \\
&\quad = -8 \int \frac{1}{u}\,du - 38 \int \frac{1}{(x - 3)^2 + 1}\,dx + \int x - 1\,dx \\
&\quad = -8 \ln|u| - 38 \tan^{-1}(x - 3) + \frac{x^2}{2} - x + C \\
&\quad = -8 \ln(x^2 - 6x + 10) - 38 \tan^{-1}(x - 3) + \frac{x^2}{2} - x + C.
\end{aligned}$$

(vi): Factor the denominator,

$$16x^4 + 8x^2 + 1 = (4x^2 + 1)^2,$$

so our Partial Fraction Decomposition looks like

$$\begin{aligned}\frac{4x^3 - 8x^2 - x + 1}{16x^4 + 8x^2 + 1} &= \frac{Ax + B}{4x^2 + 1} + \frac{Cx + D}{(4x^2 + 1)^2},\\ 4x^3 - 8x^2 - x + 1 &= (Ax + B)(4x^2 + 1) + Cx + D\\ &= 4Ax^3 + 4Bx^2 + (A + C)x + (B + D)\end{aligned}$$

This solves as

$$A = 1, \quad B = -2, \quad C = -2, \quad D = 3,$$

so that

$$\frac{4x^2 + 2x - 2}{16x^4 + 8x^2 + 1} = \frac{x - 2}{4x^2 + 1} - \frac{2x - 3}{(4x^2 + 1)^2}.$$

To integrate this, we note that

$$\int \frac{x}{4x^2+1}\,dx = \frac{\ln(4x^2+1)}{8} + C_1, \quad \text{(subst. } u = 4x^2+1\text{)}$$

$$\int \frac{1}{4x^2+1}\,dx = \frac{\tan^{-1}(2x)}{2} + C_2,$$

$$\int \frac{x}{(4x^2+1)^2}\,dx = -\frac{1}{8(4x^2+1)} + C_3, \quad \text{(subst. } u = 4x^2+1\text{)},$$

$$\int \frac{1}{(4x^2+1)^2}\,dx = \frac{\tan^{-1}(2x)}{4} + \frac{x}{2(1+x^2)} + C_4,$$

while the last one is done with the trig substitution $2x = \tan(u)$, similarly to Example 3.37 (iii). Hence,

$$\begin{aligned}\int \frac{4x^2+2x-2}{16x^4+8x^2+1}\,dx &= \int \frac{x-2}{4x^2+1} - \frac{2x-3}{(4x^2+1)^2}\,dx \\ &= \frac{\ln(4x^2+1)}{8} - 2\cdot\frac{\tan^{-1}(2x)}{2} - 2\left(-\frac{1}{8(4x^2+1)}\right) \\ &\quad + 3\left(\frac{\tan^{-1}(2x)}{4} + \frac{x}{2(1+x^2)}\right) + C \\ &= \frac{\ln(4x^2+1)}{8} - \frac{\tan^{-1}(2x)}{4} - \frac{1+6x}{4(4x^2+1)} + C.\end{aligned}$$

Here is a table summarising the integration techniques needed for different Partial Fraction Decomposition terms:

Term:	**Integration technique:**
$\frac{1}{x+a}$	$\ln \lvert x+a \rvert$.
$\frac{1}{(x+a)^k}$, $k \geq 2$	$-\frac{1}{(k-1)(x+a)^{k-1}}$.
$\frac{1}{x^2+bx+c}$, $b^2-4c<0$	Complete the square, then use $\tan^{-1}(\cdot)$.
$\frac{1}{(x^2+1)^k}$, $k \geq 2$	Trig substitution, $x = \tan(u)$.
$\frac{1}{(x^2+bx+c)^k}$, $b^2-4c<0$	Complete the square, then use a trig substitution $\cdot\cdot = \tan(u)$.
$\frac{x}{(x^2+c)^k}$, $k \geq 1$	Substitute $u = x^2 + c$.
$\frac{(2x+b)}{(x^2+bx+c)^k}$, $k \geq 1$	Substitute $u = x^2 + bx + c$.
$\frac{Ax+B}{(x^2+bx+c)^k}$, $k \geq 1$	Rewrite this as $\sum_{i=1}^{k} \frac{A_i(2x+b)+B_i}{(x^2+bx+c)^k}$, then use above.