3-8. Trigonometric substitutions

Trigonometric substitutions is one approach which is sometimes needed to solve integrals. For these, we define the substitution variable (u) implicitly:

$$x = \sin(u), \quad dx = \cos(u)\, du$$

or

$$x = \tan(u), \quad dx = \sec(u)^2\, du,$$

or

$$x = \sec(u), \quad dx = \tan(u)\sec(u)\, du.$$

Since we are defining x in terms of u (instead of the other way around as we did before with substitution), there will be an additional complication when getting the final answer in terms the original variable of integration.

Example 3.36. *Solve the following:*

(i) $$\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

(ii) $$\int \frac{x^3}{\sqrt{x^2-1}}\,dx.$$

(iii) $$\int \frac{y^2}{9y^2+1}\,dy.$$

(iv) $$\int \frac{1}{y\sqrt{2-y^2}}\,dy.$$

Solution. (i): We recognise that if $x = \sin(u)$ then $\sqrt{1-x^2} = |\cos(u)|$ (and we assume that $\cos(u) \geq 0$), so it makes sense to use

$$x = \sin(u), \quad dx = \cos(u)\,du.$$

to get

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin(u)^3}{\cos(u)} \cdot \cos(u)\,du \\
&= \int \sin(u)(1-\cos(u)^2)\,du \\
&= -\int 1 - v^2\,dv \quad \text{(subst. } v = \cos(u)\text{)} \\
&= -v + \frac{v^3}{3} + C \\
&= -\cos(u) + \frac{\cos(u)^3}{3} + C \\
&= -\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} + C.
\end{aligned}$$

(ii): Whereas in (i) we had $\sqrt{1-x^2}$ (which suggests substituting $\sin(u)$ because of the identity $\sin(u)^2+\cos(u)^2=1$), here we have $\sqrt{x^2-1}$. This suggests substituting $\sec(u)$ because of the identity $\tan(u)^2+1=\sec(x)^2$. So, we use

$$x=\sec(u), \quad dx=\sec(u)\tan(u)\,du$$

to get

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2-1}}\,dx &= \int \frac{\sec(u)^3}{\tan(u)}\cdot\sec(u)\tan(u)\,du \\
&= \int \sec(u)^4\,du \\
&= \int (\tan(u)^2+1)\sec(u)^2\,du \\
&= \int (v^2+1)\,dv \quad \text{(subst. } v=\tan(u)\text{)} \\
&= \frac{v^3}{3}+v+C \\
&= \frac{\tan(u)^3}{3}+\tan(u)+C \\
&= \frac{(x^2-1)^{3/2}}{3}+\sqrt{x^2-1}+C.
\end{aligned}$$

(iii): Observe that the numerator can be written $(3y)^2 + 1$, which suggests that we want to substitute

$$3y = \tan(u), \quad dy = \frac{\sec(u)^2}{3}\, du.$$

This leads to

$$\begin{aligned}
\int \frac{y^2}{9y^2+1}\, dy &= \int \frac{(\tan(u)/3)^2}{\tan(u)^2+1} \cdot \frac{\sec(u)^2}{3}\, du \\
&= \frac{1}{27} \int \tan(u)^2\, du \\
&= \frac{1}{27} \int \sec(u)^2 - 1\, du \\
&= \frac{1}{27} (\tan(u) - u) + C \\
&= \frac{y}{9} - \frac{1}{27} \tan^{-1}\left(\frac{y}{3}\right) + C.
\end{aligned}$$

(In the last step, we needed to invert $3y = \tan(u)$ to write u in terms of y.)

(iv): We rewrite the integrand as

$$\frac{1}{y\sqrt{2-y^2}} = \frac{1}{y\sqrt{2}\sqrt{1-(y/\sqrt{2})^2}},$$

suggesting that we want to substitute

$$\frac{y}{\sqrt{2}} = \sin(u), \quad dy = \sqrt{2}\cos(u)\,du.$$

This gives

$$\begin{aligned}\int \frac{1}{y\sqrt{2-y^2}}\,dy &= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{2}\sin(u)\cos(u)}\cdot\sqrt{2}\cos(u)\,du \\ &= \frac{1}{\sqrt{2}}\int \frac{1}{\sin(u)}\,du \\ &= \frac{1}{\sqrt{2}}\int \csc(u)\,du\end{aligned}$$

The rest of this part can be solved much like Example 3.35 (v), and is left as an exercise (in fact it is Question 1(f) on Exercise Sheet 3).

Example 3.37. *Solve the following:*

$$\text{(i)} \quad \int \frac{x}{\sqrt{4x^2+8x-5}}\,dx.$$

$$\text{(ii)} \quad \int x\sqrt{x^2-6x+10}\,dx.$$

$$\text{(iii)} \quad \int \frac{1}{(x^2+4)^2},dx.$$

Solution. (i): Whenever we have a square root of something involving x^2, it is very likely that a trigonometric substitution is needed. However, in cases like this, we need to rearrange to get something like $\sqrt{a(y^2 \pm 1)}$ (where $a \in \mathbb{R}$). This is done by "completing the square":

$$\begin{aligned} 4x^2 + 8x - 5 &= 4(x^2 + 2x + 1) - 9 \\ &= 4(x+1)^2 - 9. \end{aligned}$$

Thus,

$$\sqrt{4(x+1)^2 - 9} = 3\sqrt{\left(\frac{2(x+1)}{3}\right)^2 - 1},$$

and, as in (ii), we want to substitute

$$\frac{2(x+1)}{3} = \sec(u), \quad \frac{2}{3}\,dx = \sec(u)\tan(u)\,du.$$

It is useful to write x in terms of u before performing this substitution:

$$x = \frac{3\sec(u)}{2} - 1.$$

We get

$$
\begin{aligned}
\int \frac{x}{\sqrt{4x^2+8x-5}}\,dx &= \int \frac{x}{3\sqrt{(2(x+1)/3)^2-1}}\,dx \\
&= \int \frac{3\sec(u)/2-1}{3\tan(u)} \cdot \frac{3\sec(u)\tan(u)}{2}\,du \\
&= \frac{1}{2}\int \frac{3}{2}\sec(u)^2 - \sec(u)\,du \\
&= \frac{3}{4}\tan(u) - \frac{1}{2}\ln\left|\sec(u)+\tan(u)\right| + C \\
&= \frac{3}{4}\sqrt{\left(\frac{2(x+1)}{3}\right)^2-1} \\
&\qquad - \frac{1}{2}\ln\left|\frac{2(x+1)}{3}+\sqrt{\left(\frac{2(x+1)}{3}\right)^2-1}\right| + C \\
&= \frac{1}{4}\sqrt{4(x+1)^2-9} \\
&\qquad - \frac{1}{2}\left(\ln\left|2(x+1)+\sqrt{4(x+1)^2-9}\right| - \ln(3)\right) + C \\
&= \frac{1}{4}\sqrt{4(x+1)^2-9} \\
&\qquad - \frac{1}{2}\ln\left|2(x+1)+\sqrt{4(x+1)^2-9}\right| + C'.
\end{aligned}
$$

(ii): Completing the square, we have

$$\sqrt{x^2 - 6x + 10} = \sqrt{(x-3)^2 + 1},$$

so we use the substitution

$$x - 3 = \tan(u), \quad dx = \sec(u)^2\, du.$$

This yields

$$\begin{aligned}\int x\sqrt{x^2 - 6x + 10}\, dx &= \int (\tan(u) + 3)^2 \sec(u) \cdot \sec(u)^2\, du \\ &= \int \tan(u)\sec(u)^3 + 3\sec(u)^3\, du.\end{aligned}$$

We break this into two integrals and solve them separately. For the first one, substitute

$$v = \sec(u), \quad dv = \tan(u)\sec(u)\, du$$

to get

$$\begin{aligned}\int \tan(u)\sec(u)^3\, du &= \int v^2\, dv \\ &= \frac{v^3}{3} + C \\ &= \frac{\sec(u)^3}{3} + C.\end{aligned}$$

For the second one, Example 3.35 (vi) tells us that

$$\int \sec(u)^3 \, du = \frac{1}{2}(\sec(u)\tan(u) + \ln|\sec(u) + \tan(u)|) + C'.$$

We therefore have

$$\begin{aligned}\int x\sqrt{x^2 - 6x + 10}\, dx &= \int \tan(u)\sec(u)^3 + 3\sec(u)^3 \, du \\ &= \frac{\sec(u)^3}{3} + \frac{\sec(u)\tan(u) + \ln|\sec(u) + \tan(u)|}{2} + C'' \\ &= \frac{((x-3)^2+1)^{3/2}}{3} + \frac{\sqrt{(x-3)^2+1}(x-3)}{2} \\ &\quad + \frac{\ln\left|\sqrt{(x-3)^2+1} + (x-3)\right|}{2} + C''.\end{aligned}$$

(iii): Write the integrand as

$$\frac{1}{(x^2+4)^2} = \frac{1}{16((x/2)^2+1)^2},$$

and then use the substitution

$$x/2 = \tan(u), \quad dx = 2\sec(u)^2\,dx.$$

This leads to

$$\begin{aligned}
\int \frac{1}{(x^2+4)^2}\,, dx &= \int \frac{1}{16((x/2)^2+1)^2}\,dx \\
&= \int \frac{2\sec(u)^2}{16(\tan(u)^2+1)^2}\,du \\
&= \frac{1}{8}\int \frac{\sec(u)^2}{\sec(u)^4}\,du \\
&= \frac{1}{8}\int \cos(u)^2\,du \\
&= \frac{1}{16}\int 1+\cos(2u)\,du \\
&= \frac{1}{16}\left(u+\frac{\sin(2u)}{2}\right)+C \\
&= \frac{1}{16}\left(u+\sin(u)\cos(u)\right)+C \\
&= \frac{1}{16}\left(u+\frac{\tan(u)}{\sec(u)^2}\right)+C \\
&= \frac{1}{16}\left(\tan^{-1}\left(\frac{x}{2}\right)+\frac{x/2}{x^2/4+1}\right)+C. \\
&= \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right)+\frac{x}{8(x^2+4)}+C.
\end{aligned}$$

As we saw in the previous examples, the general rules for trig substitutions are:

If the integrand contains ...	Substitute ...
$1 - x^2$ (especially $\sqrt{1 - x^2}$)	$x = \sin(u)$
$x^2 - 1$ (especially $\sqrt{x^2 - 1}$)	$x = \sec(u)$
$x^2 + 1$ (especially $\sqrt{x^2 + 1}$)	$x = \tan(u)$.

When the integrand contains a quadratic form (especially under a square root), first complete the square to get it in one of the above forms.