**Example 3.33.** *Solve the following integrals.*

*(i)* $$\int \sin(x)\cos(x)^7\,dx.$$

*(ii)* $$\int \cos(x)^7\,dx.$$

*(iii)* $$\int \cos(x)^4\sin(x)^5\,dx.$$

**Solution.** (i): We have already seen problems like this in Section 3 (Substitution Rule). We substitute

$$u = \cos(x), \quad du = -\sin(x)\,dx$$

to get

$$\begin{aligned} \int \sin(x)\cos(x)^7\,dx &= -\int u^7\,du \\ &= -\frac{u^8}{8} + C \\ &= -\frac{\cos(x)^8}{8} + C. \end{aligned}$$

(ii): We want to make a substitution as in (i), but at first this doesn't seem to have the right form. Using the identity

$$\cos(x)^2 + \sin(x)^2 = 1,$$

we get

$$\cos(x)^7 = (1 - \sin(x)^2)^3 \cos(x),$$

and thus the substitution

$$u = \sin(x), \quad du = \cos(x)\,dx$$

will work. We compute

$$\begin{aligned}
\int \cos(x)^7\,dx &= \int (1 - \sin(x)^2)^3 \cos(x)\,dx \\
&= \int (1 - u^2)^3\,du \\
&= \int 1 - 3u^2 + 3u^4 - u^6\,du \\
&= u - u^3 + \frac{3u^5}{5} - \frac{u^7}{7} + C \\
&= \sin(x) - \sin(x)^3 + \frac{3\sin(x)^5}{5} - \frac{\sin(x)^7}{7} + C.
\end{aligned}$$

(iii): Again we use the identity $\sin(x)^2 + \cos(x)^2 = 1$, and then substitute

$$u = \cos(x), \quad du = -\sin(x)\,dx,$$

to get

$$\begin{aligned}
\int \cos(x)^4 \sin(x)^5\,dx &= \int \cos(x)^4 (1 - \cos(x)^2)^2 \sin(x)\,dx \\
&= -\int u^4 (1-u^2)^2\,du \\
&= \int -u^4 + 2u^6 - u^8\,du \\
&= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C \\
&= -\frac{\cos(x)^5}{5} + \frac{2\cos(x)^7}{7} - \frac{\cos(x)^9}{9} + C.
\end{aligned}$$

More generally, using the identity

$$\sin(x)^2 + \cos(x)^2 = 1$$

and substitution, we can solve

$$\int \sin(x)^m \cos(x)^n \, dx$$

provided that $m$ and $n$ are positive integers, and one of $m$ or $n$ is odd. What if both $m$ and $n$ are even? To handle these, we will need to make use of angle-sum formulas from trigonometry; we recall these are

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B)),$$

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B)).$$

It is helpful to remember the special cases where $A = B$:

$$\sin(A)^2 = \frac{1}{2}(1 - \cos(2A)),$$
$$\cos(A)^2 = \frac{1}{2}(1 + \cos(2A))),$$
$$\sin(A)\cos(A) = \frac{1}{2}\sin(2A).$$

With these, we can reduce any expression of the form

$$\sin(x)^m \cos(x)^n$$

into something involving only single powers of $\sin(kx)$ and $\cos(kx)$ (for various values of $k$). In fact, we can even handle *any* product of terms of the form $\sin(kx)$ and $\cos(kx)$.

**Example 3.34.** *Solve the following integrals.*

*(i)* $$\int \sin(x)^2 \cos(x)^2 \, dx.$$

*(ii)* $$\int \sin(x) \sin(\sqrt{2}x) \sin(\sqrt{3}x) \, dx.$$

*(iii)* $$\int \sin(x)^2 \cos(5x)^2 \, dx.$$

**Solution.** (i): Start by reducing the integrand,

$$\begin{aligned}
\sin(x)^2\cos(x)^2 &= \sin(x)^2(1-\sin(x)^2)\\
&= \sin(x)^2 - \sin(x)^4\\
&= \frac{1}{2}(1-\cos(2x)) - \left(\frac{1}{2}(1-\cos(2x))\right)^2\\
&= \frac{1}{2} - \frac{\cos(2x)}{2} - \frac{1}{4} + \frac{\cos(2x)}{2} - \frac{\cos(2x)^2}{4}\\
&= \frac{1}{4} - \frac{1}{4}\cdot\frac{1}{2}(\cos(4x)+1)\\
&= \frac{1}{8} - \frac{\cos(4x)}{8}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \sin(x)^2\cos(x)^2\,dx &= \int \frac{1}{8} - \frac{\cos(4x)}{8}\,dx\\
&= \frac{x}{8} - \frac{\sin(4x)}{4\cdot 8} + C.
\end{aligned}$$

(ii): Simplify the integrand:

$$\begin{aligned}\sin(x)\sin(\sqrt{2}x)\sin(\sqrt{3}x) &= \frac{1}{2}(\cos(x-\sqrt{2}x)-\cos(x+\sqrt{2}x))\sin(\sqrt{3}x)\\ &= \frac{1}{4}(\sin(x-\sqrt{2}x+\sqrt{3}x)-\sin(x-\sqrt{2}x-\sqrt{3}x)\\ &\qquad -\sin(x+\sqrt{2}x+\sqrt{3}x)+\sin(x+\sqrt{2}x-\sqrt{3}x)),\end{aligned}$$

Hence,

$$\begin{aligned}\int \sin(x)\sin(\sqrt{2}x)\sin(\sqrt{3}x)\,dx &= \frac{1}{4}\int \sin(x-\sqrt{2}x+\sqrt{3}x)-\sin(x-\sqrt{2}x-\sqrt{3}x)\\ &\qquad -\sin(x+\sqrt{2}x+\sqrt{3}x)+\sin(x+\sqrt{2}x-\sqrt{3}x)\,dx\\ &= \frac{1}{4}\left(-\frac{\cos((1-\sqrt{2}+\sqrt{3})x)}{1-\sqrt{2}+\sqrt{3}}\right.\\ &\qquad +\frac{\cos((1-\sqrt{2}-\sqrt{3})x)}{1-\sqrt{2}-\sqrt{3}}\\ &\qquad +\frac{\cos((1+\sqrt{2}+\sqrt{3})x)}{1+\sqrt{2}+\sqrt{3}}\\ &\qquad \left.-\frac{\cos((1+\sqrt{2}-\sqrt{3})x)}{1+\sqrt{2}-\sqrt{3}}\right)\end{aligned}$$

(iii): Reduce the integrand,

$$\begin{aligned}\sin(x)^2\cos(5x)^2 &= \frac{1}{4}(1-\cos(2x))(1+\cos(10x)) \\ &= \frac{1}{4}(1-\cos(2x)+\cos(10x)-\cos(2x)\cos(10x)) \\ &= \frac{1}{4}(1-\cos(2x)+\cos(10x)) - \frac{1}{8}(\cos(12x)+\cos(8x)).\end{aligned}$$

Thus,

$$\begin{aligned}\int \sin(x)^2\cos(5x)^2\,dx &= \frac{1}{4}\int 1-\cos(2x)+\cos(10x)\,dx \\ &\quad - \frac{1}{8}\int \cos(12x)+\cos(8x)\,dx \\ &= \frac{1}{4}\left(x-\frac{\sin(2x)}{2}+\frac{\sin(10x)}{10}\right) \\ &\quad - \frac{1}{8}\left(\frac{\sin(12x)}{12}+\frac{\sin(8x)}{8}\right)+C.\end{aligned}$$

What if we have something of the form $\int \sin(x)^m \cos(x)^n \, dx$, where one (or both) of $m, n$ are negative integers?
It is often helpful as a first step to rewrite the integrand in terms of

$$\sec(x) = \cos(x)^{-1}, \quad \csc(x) = \sin(x)^{-1},$$
$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}.$$

(Often the question will already be in terms of these functions.) The identities

$$\tan(x)^2 + 1 = \sec(x)^2,$$
$$\cot(x)^2 + 1 = \csc(x)^2$$

are useful. Also, common substitutions in these cases are

$$u = \tan(x), \quad du = \sec(x)^2 \, dx,$$
$$u = \sec(x), \quad du = \sec(x)\tan(x) \, dx,$$
$$u = \cot(x), \quad du = -\csc(x)^2 \, dx, \quad \text{or}$$
$$u = \csc(x), \quad du = -\csc(x)\cot(x) \, dx.$$

**Example 3.35.** *Solve the following integrals:*

*(i)* $$\int \frac{\sin(x)^5}{\cos(x)^4}\,dx.$$

*(ii)* $$\int \frac{\sin(x)^4}{\cos(x)^6}\,dx.$$

*(iii)* $$\int \tan(x)\,dx.$$

*(iv)* $$\int \tan(x)^2\,dx.$$

*(v)* $$\int \sec(x)\,dx.$$

*(vi)* $$\int \sec(x)^3\,dx.$$

**Solution.** (i): We have $\sin(x)$ to an odd power, so we can use the substitution

$$u = \cos(x), \quad du = -\sin(x)\,dx.$$

With this we get

$$\begin{aligned}
\int \frac{\sin(x)^5}{\cos(x)^4}\,dx &= \int \sin(x) \cdot \frac{(1-\cos(x)^2)^2}{\cos(x)^4}\,dx \\
&= -\int \frac{(1-u^2)^2}{u^4}\,du \\
&= -\int \frac{1}{u^4} - \frac{2}{u^2} + 1\,du \\
&= \frac{1}{3u^3} - \frac{2}{u} - u + C \\
&= \frac{1}{3\cos(x)^3} - \frac{2}{\cos(x)} - \cos(x) + C.
\end{aligned}$$

(ii): Here we don't have an odd power on either $\sin(x)$ or $\cos(x)$. However, we may rewrite the integrand as

$$\tan(x)^4 \sec(x)^2,$$

which suggests we could try to substitute either $u = \tan(x)$ or $u = \sec(x)$. Since $\tan(x)$ has an even exponent, we can't substitute $u = \sec(x)$, so we use

$$u = \tan(x), \quad du = \sec(x)^2\, dx.$$

This yields

$$\begin{aligned}
\int \frac{\sin(x)^4}{\cos(x)^6}\, dx &= \int \tan(x)^4 \sec(x)^2\, dx \\
&= \int u^4\, du \\
&= \frac{u^5}{5} + C \\
&= \frac{\tan(x)^5}{5} + C.
\end{aligned}$$

(iii): We rewrite $\tan(x) = \sin(x)/\cos(x)$ and then notice that the best substitution is

$$u = \cos(x), \quad du = -\sin(x)\,dx.$$

Thus,

$$\begin{aligned} \int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\,dx \\ &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\cos(x)| + C. \end{aligned}$$

(iv): This one is just a little trick. Using the identity $\tan(x)^2 + 1 = \sec(x)^2$, one gets

$$\int \tan(x)^2 \, dx = \int \sec(x)^2 - 1 \, dx = \tan(x) - x + C.$$

(v): This one requires a manipulation of the integrand that isn't obvious.

$$\sec(x) = \frac{\sec(x)(\sec(x) + \tan(x))}{\sec(x) + \tan(x)} = \frac{\sec(x)^2 + \sec(x)\tan(x)}{\sec(x) + \tan(x)}.$$

The numerator is the derivative of the denominator, so this is now amenable to the substitution

$$u = \sec(x) + \tan(x), \quad du = (\sec(x)\tan(x) + \sec(x)^2)\, dx.$$

We get

$$\begin{aligned} \int \sec(x)\, dx &= \int \frac{\sec(x)^2 + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\, dx \\ &= \int \frac{1}{u}\, du \\ &= \ln|u| + C \\ &= \ln|\sec(x) + \tan(x)| + C. \end{aligned}$$

(vi): Whether we write the integrand as $\sec(x)^3$ or $1/\cos(x)^3$, it seems that there is no substitution that will help. Instead we do integration by parts, with

$$u = \sec(x), \quad dv = \sec(x)^2\,dx$$
$$du = \sec(x)\tan(x)\,dx, \quad v = \tan(x).$$

With this we get

$$\int \sec(x)^3\,dx = \sec(x)\tan(x) - \int \tan(x)^2\sec(x)\,dx \quad \text{(IBP)}.$$

At first this might not look like it helps, because we equally can't solve

$$\int \tan(x)^2\sec(x)\,dx = \int \frac{\sin(x)^2}{\cos(x)^3}\,dx$$

using substitutions as before.

The correct next step is to rewrite $\tan(x)^2$ in terms of $\sec(x)$:

$$\begin{aligned}\int \sec(x)^3\,dx &= \sec(x)\tan(x) - \int \tan(x)^2 \sec(x)\,dx \\ &= \sec(x)\tan(x) - \int (\sec(x)^2 - 1)\sec(x)\,dx \\ &= \sec(x)\tan(x) - \int \sec(x)^3\,dx + \int \sec(x)\,dx \\ &= \sec(x)\tan(x) - \int \sec(x)^3\,dx + \ln|\sec(x)+\tan(x)| + C,\end{aligned}$$

where we used (v) to get $\int \sec(x)\,dx$. Now we rearrange to solve for $\int \sec(x)^3\,dx$:

$$\begin{aligned}2\int \sec(x)^3\,dx &= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| + C, \\ \int \sec(x)^3\,dx &= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|) + C'.\end{aligned}$$