3-6. Integration by parts

Integration by parts (IBP) is a new technique, which will enable us to solve integrals such as

$$\int xe^{6x}\,dx,$$

which cannot be solved by substitution.

Proposition 3.23 (Integration by parts). *Let $f(x), g(x)$ be differentiable functions, and suppose that*

$$\int f(x)g'(x)\,dx = H(x) + C.$$

Then

$$\int f'(x)g(x)\,dx = f(x)g(x) - H(x) + C'.$$

Proof.

This is derived from the product rule for differentiation. By assumption, we know that

$$H'(x) = f(x)g'(x).$$

Define $F(x) = f(x)g(x) - H(x)$, so that

$$F'(x) = f'(x)g(x) + f(x)g'(x) - f(x)g'(x) = f'(x)g(x),$$

i.e.,

$$\int f'(x)g(x)\,dx = F(x) + C',$$

as required. $\square$

Here is a more concise way of remembering Integration by Parts. Use

$$\begin{aligned} u &= f(x), & dv &= g'(x)\,dx \\ du &= f'(x),\,dx & v &= g(x) \end{aligned}$$

and then IBP becomes

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Since a new integration constant will occur for the integral $\int v\,du$, we have dropped the integration constant ("$+\,C$").

Example 3.24. *Solve*

$$\int xe^{6x}\,dx.$$

Solution. Use

$$u = x, \quad dv = e^{6x}\,dx$$
$$du = dx, \quad v = \frac{1}{6}e^{6x}$$

to get

$$\begin{aligned}\int xe^{6x} &= \int u\,dv \\ &= uv - \int v\,du \qquad \text{(IBP)} \\ &= \frac{1}{6}\left(xe^{6x} - \int e^{6x}\,dx\right) \\ &= \frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + C\end{aligned}$$

Check:

$$\frac{d}{dx}(\frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + C) = \frac{1}{6}(e^{6x} + 6xe^{6x}) - \frac{1}{36}6e^{6x} = xe^{6x}.$$

How did we choose u and dv in the above example?

Generally, we want to factor the integrand as $u\,dv$ where:

1. we can find the antiderivative to dv (i.e., find v), and
2. the integral $\int v\,du$, is **easier** to solve.

(While this is generally what we want to do, there is one trick where 2 doesn't hold – $\int v\,du$ isn't any simpler; see Example 3.31.)

In the above example, it helped to remember that we knew how to integrate e^x (and, by substation, also e^{6x}).

Through **practice** with computing integrals, we get to know which things we can integrate more easily.

It might be that we need to use substitution, or even IBP again, to find v or solve $\int v\,du$.

Example 3.25. *Evaluate the integral*

$$\int x\sqrt{x+1}\,dx$$

Solution. There are, in fact, two ways of doing this one: (a) Integration by Parts, and (b) Substitution.

(a) Integration by Parts. Notice that there are no trigonometric or exponential functions here. Although often the IBP integrals we will be solving contain trig or exponential functions, **don't assume** that you can't use IBP if you don't see them. We'll use IBP with

$$u = x, \quad dv = \sqrt{x+1}\,dx$$
$$du = dx, \quad v = \frac{2}{3}(x+1)^{3/2}.$$

Then

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}\,dx \qquad \text{(IBP)}\\ &= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(x+1)^{5/2} + C\\ &= \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C\end{aligned}$$

(b) Substitution. We use the following substitution

$$u = x + 1, \quad x = u - 1, \quad dx = du.$$

Then

$$\begin{aligned}
\int x\sqrt{x+1}\,dx &= \int (u-1)u^{1/2}\,du \\
&= \int u^{3/2} - u^{1/2}\,du \\
&= \frac{5}{2}u^{5/2} - \frac{3}{2}u^{3/2} + C' \\
&= \frac{5}{2}(x+1)^{5/2} - \frac{3}{2}(x+1)^{3/2} + C'
\end{aligned}$$

We got different answers! Did we do something wrong? No, in fact, with some algebraic manipulation, we can show that

$$\frac{5}{2}(x+1)^{5/2} - \frac{3}{2}(x+1)^{3/2} = \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2}.$$

(Note, in some cases, there might be two correct solutions that aren't exactly the same, but instead differ by a constant. This is also okay — it is precisely the purpose of the integration constant.)

Example 3.26. *Solve*

$$\int x^2 \sin(10x)\, dx$$

Solution. We use IBP with

$$u = x^2, \quad dv = \sin(10x)\, dx$$
$$du = 2x\, dx, \quad v = -\frac{1}{10}\cos(10x).$$

Then

$$\int x^2 \sin(10x)\, dx = -\frac{1}{10}\left(x^2\cos(10x) - 2\int x\cos(10x)\, dx\right) \qquad \text{(IBP)}$$

At first, this might not seem helpful, since the new integral, $\int x\cos(10x)\, dx$, is still not something we recognise. We need to do IBP again, this time with

$$u = x, \quad dv = \cos(10x)\, dx$$
$$du = dx, \quad v = \frac{1}{10}\sin(10x).$$

Then

$$\int x\cos(10x)\,dx = \frac{1}{10}\left(x\sin(10x) - \int \sin(10x)\,dx\right) \qquad \text{(IBP)}$$
$$= \frac{1}{10}x\sin(10x) + \frac{1}{100}\cos(10x) + C$$

and thus,

$$\int x^2\sin(10x)\,dx = -\frac{1}{10}x^2\cos(10x) + \frac{1}{50}x\sin(10x) + \frac{1}{500}\cos(10x) + C'$$

Example 3.27. *Evaluate*

$$\int \ln(x)\, dx.$$

Solution. Unlike in previous examples, the integrand doesn't look like it can factor at all! One choice that you might be tempted to try is

$$u = 1, \quad dv = \ln(x)\, dx.$$

However, this begs the question, because we don't know how to find v (i.e., antidifferentiate $\ln(x)$) in the first place!
Instead, we use

$$u = \ln(x), \quad dv = dx$$
$$du = \frac{1}{x}\, dx, \quad v = x.$$

Then

$$\begin{aligned} \int \ln(x)\, dx &= x\ln(x) - \int x \frac{1}{x}\, dx \qquad \text{(IBP)} \\ &= x\ln(x) - \int 1\, dx \\ &= x\ln(x) - x + C. \end{aligned}$$

Example 3.28. *Solve*

$$\int \ln(t)^2\,dt.$$

Solution. From the previous example, we know an antiderivative to $\ln(t)$. We can use

$$u = \ln(t), \quad dv = \ln(t)\,dt$$
$$du = \frac{dt}{t}, \quad v = t\ln(t) - t,$$

to get

$$\begin{aligned}\int \ln(t)^2\,dt &= \ln(t)(t\ln(t) - t) - \int \frac{t\ln(t) - t}{t}\,dt \quad \text{(IBP)}\\ &= t\ln(t)^2 - t\ln(t) - \int \ln(t) - 1\,dt\\ &= t\ln(t)^2 - t\ln(t) - (t\ln(t) - t - t) + C\\ &= t\ln(t)^2 - 2t\ln(t) + 2t + C.\end{aligned}$$

This one could also have been done with $u = \ln(t)^2, dv = 1\,dt$.

Example 3.29. *Solve the integral*

$$\int x^5 \sqrt{x^3+1}\,dx$$

Solution. The most obvious way to factorise the integrand is

$$u = x^5, \quad dv = \sqrt{x^3+1}\,dx.$$

To find v, however, requires solving

$$\int \sqrt{x^3+1}\,dx$$

which is not ~~easy~~ possible!

But this is not the only choice! There are many others, e.g., $x \cdot x^4\sqrt{x^3+1}$, $x^2 \cdot x^3\sqrt{x^3+1}$, etc. The best way to do this one is by choosing $dv = x^2\sqrt{x^3+1}$. This is because we can do a substitution to solve

$$\int x^2\sqrt{x^3+1}\,dx,$$

namely $w = x^3+1$, $dw = 3x^2\,dx$, which gives

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int \sqrt{w}\,dw = \frac{2}{9}(x^3+1)^{3/2} + C.$$

Thus, we use

$$u = x^3, \quad dv = x^2\sqrt{x^3+1},$$
$$du = 3x^2, \quad v = \frac{2}{9}(x^3+1)^{3/2}$$

and get

$$\begin{aligned}
\int x^5\sqrt{x^3+1}\,dx &= \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{2}{3}\int x^2(x^3+1)^{3/2}\,dx \qquad \text{(IBP)} \\
&= \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{2}{9}\int w^{3/2}\,dw \qquad \text{(subst. } w = x^3+1\text{)} \\
&= \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{4}{45}w^{5/2} + C \\
&= \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{4}{45}(x^3+1)^{5/2} + C.
\end{aligned}$$

Example 3.30. *Evaluate*

$$\text{(i)} \quad \int x^3 e^{x/2}\, dx.$$

$$\text{(ii)} \quad \int \frac{\ln(t)^2}{t^2}\, dt.$$

Solution. (i): Use IBP with

$$u = x^3, \quad dv = e^{x/2}\,dx$$
$$du = 3x^2\,dx, \quad v = 2e^{x/2}$$

to get

$$\int x^3 e^{x/2}\,dx = 2x^3 e^{x/2} - 6\int x^2 e^{x/2}\,dx.$$

Does this help? Well the new integral isn't something we can solve directly; but it looks less complicated (because it contains x^2 instead of x^3). We simply need to use IBP again (and again). This time use

$$u = x^2, \quad dv = e^{x/2}\,dx$$
$$du = 2x\,dx, \quad v = 2e^{x/2},$$

which gives

$$\int x^2 e^{x/2}\,dx = 2x^2 e^{x/2} - 4\int x e^{x/2}\,dx \quad \text{(IBP)}.$$

We use IBP again; this time,

$$u = x, \quad dv = e^{x/2}\,dx$$
$$du = 1\,dx, \quad v = 2e^{x/2},$$

which gives

$$\begin{aligned}\int xe^{x/2}\,dx &= 2xe^{x/2} - 2\int e^{x/2}\,dx \quad \text{(IBP)}\\ &= 2xe^{x/2} - 4e^{x/2} + C.\end{aligned}$$

Altogether, we get

$$\begin{aligned}\int x^3e^{x/2}\,dx &= 2x^3e^{x/2} - 6[2x^2e^{x/2} - 4\left(2xe^{x/2} - 4e^{x/2} + C\right)]\\ &= 2e^{x/2}(x^3 - 6x^2 + 24x - 48) + C'.\end{aligned}$$

(ii): Use

$$u = \ln(t)^2, \quad dv = \frac{1}{t^2}\,dt$$
$$du = \frac{2\ln(t)}{t}\,dt, \quad v = -\frac{1}{t}.$$

Then IBP gives

$$\int \frac{\ln(t)^2}{t^2}\,dt = -\frac{\ln(t)^2}{t} + 2\int \frac{\ln(t)}{t^2}\,dt \quad \text{(IBP)}.$$

The situation is much like in (i): we arrive at a new integral that we can't solve directly, but it seems that we are getting closer to something that we can solve directly, and so we push on.

We now use

$$u = \ln(t), \quad dv = \frac{1}{t^2}\,dt$$
$$du = \frac{1}{t}\,dt, \quad v = -\frac{1}{t}$$

to get

$$\int \frac{\ln(t)}{t^2}\,dt = -\frac{\ln(t)}{t} + \int \frac{1}{t^2}\,dt \quad \text{(IBP)}$$
$$= -\frac{\ln(t)}{t} - \frac{1}{t} + C.$$

Putting these together we obtain

$$\begin{aligned}\int \frac{\ln(t)^2}{t^2}\,dt &= -\frac{\ln(t)^2}{t} + 2\int \frac{\ln(t)}{t^2}\,dt \\ &= -\frac{\ln(t)^2}{t} - \frac{2\ln(t)}{t} - \frac{2}{t} + C'.\end{aligned}$$

In the previous example, it was clear that we should continue solving the integral using IBP, since each time the new integral appearing became easier. In the next example, this will not be the case; it involves a trick.

Example 3.31. *Evaluate*

$$\int e^t \cos(t)\, dt.$$

Solution. Try

$$u = e^t, \quad dv = \cos(t)\, dt,$$
$$du = e^t\, dt, \quad v = \sin(t),$$

and we find

$$\int e^t \cos(t)\, dt = e^t \sin(t) - \int e^t \sin(t)\, dt \qquad \text{(IBP)}. \tag{2}$$

The new integral, $\int e^t \sin(t)\, dt$, looks no easier than the original!

We might have instead tried

$$u = \cos(t), \quad dv = e^t\,dt,$$
$$du = -\sin(t)\,dt, \quad v = e^t,$$

which gives

$$\int e^t\cos(t)\,dt = e^t\cos(t) + \int e^t\sin(t)\,dt \qquad \text{(IBP)}. \tag{3}$$

This still leads to something involving the integral $\int e^t\sin(t)\,dt$. It seems that there is no way to get around something involving this other integral.
So let's perservere and try

$$u = \sin(t), \quad dv = e^t\,dt,$$
$$du = \cos(t)\,dt, \quad v = e^t,$$

which gives

$$\int e^t\sin(t)\,dt = e^t\sin(t) - \int e^t\cos(t)\,dt \qquad \text{(IBP)},$$

which expresses the new integral $\int e^t\sin(t)\,dt$ in terms of the original integral.

Does this help? Well, combining this with either (2) or (3) leads to an equation involving only the original integral. With (2), we get

$$\begin{aligned}
\int e^t \cos(t)\,dt &= e^t \sin(t) - \int e^t \sin(t)\,dt \\
&= e^t \sin(t) - (e^t \sin(t) - \int e^t \cos(t)\,dt) \\
&= \int e^t \cos(t)\,dt.
\end{aligned}$$

This is not helpful at all — what has happened was that second IBP undid the first IBP.

However, with (3), we get

$$\int e^t \cos(t)\,dt = e^t \cos(t) + \int e^t \sin(t)\,dt$$
$$= e^t \cos(t) + e^t \sin(t) - \int e^t \cos(t)\,dt$$

and rearranging, this becomes

$$2\int e^t \cos(t) = e^t \cos(t) + e^t \sin(t),$$

or

$$\int e^t \cos(t) = \frac{1}{2} e^t (\cos(t) + \sin(t)).$$

Note that there should be a constant of integration, so we add it, to get

$$\int e^t \cos(t) = \frac{1}{2} e^t (\cos(t) + \sin(t)) + C.$$

(There should **always** be a constant of integration when solving an indefinite integral; the only reason it wasn't there in our original solution is that we've established a habit of not adding one every time we do IBP, because usually it appears later when we get to an integral we can solve.)

Example 3.32. *Solve the following.*

$$\text{(i)} \quad \int \cos(\sqrt{1-y})\, dy.$$

$$\text{(ii)} \quad \int e^x \sin^{-1}(e^x)\, dx.$$

Solution. (i): Initially it looks like there isn't much that can be done here. In fact, we want to start with the substitution

$$w = \sqrt{1-y}, \quad dw = -\frac{1}{2\sqrt{1-y}}\,dy = -\frac{1}{2w}\,dy$$

(Note that we could have equally done this as

$$w^2 = 1-y, \quad 2w\,dw = -1\,dy.)$$

This gives

$$\int \cos(\sqrt{1-y})\,dy = \int -2w\cos(w)\,dw,$$

and this looks like something we can handle using IBP (we've done something very similar in Example 3.26).
We set

$$\begin{aligned} u &= w, \quad dv = \cos(w)\,dw \\ du &= dw, \quad v = \sin(w) \end{aligned}$$

and use this with IBP, to get

$$\begin{aligned} \int \cos(\sqrt{1-y})\,dy &= -2\int w\cos(w)\,dw \\ &= -2\left(w\sin(w) - \int \sin(w)\,dw\right) \quad \text{(IBP)} \\ &= -2\,(w\sin(w) + \cos(w)) \\ &= -2(\sqrt{1-y}\sin(\sqrt{1-y}) + \cos(\sqrt{1-y})). \end{aligned}$$

(ii): Again, we start with a substitution:

$$w = e^x, \quad dw = e^x \, dx,$$

which gives

$$\int e^x \sin^{-1}(e^x) \, dx = \int \sin^{-1}(w) \, dw.$$

Now we do IBP with

$$u = \sin^{-1}(w), \quad dv = 1 \, dw$$

$$du = \frac{1}{\sqrt{1-w^2}} \, dw, \quad v = w.$$

In the following, we will find we need to do another substitution:

$$\begin{aligned}
\int e^x \sin^{-1}(e^x) \, dx &= \int \sin^{-1}(w) \, dw \\
&= w \sin^{-1}(w) - \int \frac{w}{\sqrt{1-w^2}} \, dw \quad \text{(IBP)} \\
&= w \sin^{-1}(w) - \int -\frac{1}{2\sqrt{z}} \, dz \quad (\text{subst. } z = 1 - w^2) \\
&= w \sin^{-1}(w) + 2 \cdot \frac{1}{2}\sqrt{z} + C \\
&= w \sin^{-1}(w) + \sqrt{1-w^2} + C \\
&= e^x \sin^{-1}(e^x) + \sqrt{1-e^{2x}} + C.
\end{aligned}$$

Summary of IBP techniques

(a) **IBP to reduce a power of the integration variable**. If the integrand factors as $x^k g(x)$, and we know how to integrate $g(x)$, then we set $u = x^k$ and $v = g(x)$ (Examples 3.24, 3.25). When $k > 1$, we will probably need to do this again (and again ...) (Examples 3.26, 3.30 (i)).

(b) **IBP with** $u \neq x^k$. Sometimes we don't have a factor of x^k, or we do but we can't integrate the other factor. Then we need to try something else. Don't forget to try $dv = 1\,dx$. (Examples 3.27, 3.28, 3.29, 3.30 (ii).)

(c) **IBP twice, returning something involving the original integral**. Provided the second IBP didn't undo the first one, we get an equation which we can solve, yielding a solution to the original integral (Example 3.31).

(d) **Combining substitution and IBP**. Example 3.32 (i), which starts with substitution then uses (a). Example 3.32 (ii), which starts with substitution and then uses (b), and finishes with another substitution.