3-5. The Fundamental Theorem of Calculus

In general, it can be very tedious to compute a definite integral using Riemann sums. The following makes it much easier.

Theorem 3.16 (Fundamental Theorem of Calculus). *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function.*
(i) For each $x \in [a, b]$*, define*

$$F(x) = \int_a^x f(t)\,dt.$$

Then

$$\int f(x)\,dx = F(x) + C$$

(i.e., $F(x)$ *is differentiable and* $F'(x) = f(x)$*).*

(ii) If $G(x)$ *is any antiderivative of* $f(x)$ *then*

$$\int_a^b f(x)\,dx = G(x)\big|_{x=a}^{b},$$

where $G(x)\big|_{x=a}^{b}$ *means* $G(b) - G(a)$*.*
Again, we won't discuss the proof of this theorem, leaving it for MA2509 "Analysis II".

The purpose of this theorem is two-fold.

Part (i) tells us that **every continuous function** has an antiderivative (even though we may not have a closed form expression).

Part (ii) (which is more important in this course) tells us how to use indefinite integrals to compute definite integrals. Concisely, it says

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Bigg|_{x=a}^{b}.$$

One interpretation of the integral is as a way of defining the average value of a continuous function: if $f(x)$ is a continuous function, then its average value on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1}$$

Consider the case that the function $f(t)$ represents the speed of a car, at time t. If $p(t)$ represents the position of the car, then $p'(t) = f(t)$.

Two ways of measuring the average speed between times a and b are: divide the total distance travelled by the total time:

$$\frac{p(b) - p(a)}{b - a},$$

or use the integral formula (1)

$$\frac{1}{b-a} \int_a^b f(t)\, dt.$$

Since $p'(t) = f(t)$, FTC tells us that these two ways of measuring the average are the same:

$$\frac{p(b) - p(a)}{b - a} = \frac{1}{b-a} \int_a^b p'(t)\, dt = \frac{1}{b-a} \int_a^b f(t)\, dt.$$

Example 3.17. *Compute the area enclosed by the x-axis and the curve* $y = 4 - x^2$.

Solution. *Note that the curve* $y = 4 - x^2$ *intersects the x-axis at the points* $(-2, 0)$ *and* $(2, 0)$. *The area we need to compute is thereby given by the integral*

$$\int_{-2}^{2} 4 - t^2 \, dt.$$

We compute

$$\int 4 - t^2 \, dt = 4t - \frac{t^3}{3} + C,$$

so that

$$\int_{-2}^{2} 4 - t^2 \, dt = 4t - \frac{t^3}{3}\bigg|_{t=-2}^{2} = 8 - \frac{8}{3} - (-8 + \frac{8}{3}) = \frac{32}{3}.$$

When computing a definite integral using FTC, we always drop the integration constant (i.e., the "$+C$"). (If we left it in, it would cancel with itself.)

Example 3.18. *Differentiate the following functions.*

$$\text{(i)} \quad f(x) = \int_5^x e^{7t^2} \sqrt{5 + \cos(t)^3}\, dt$$

$$\text{(ii)} \quad g(x) = \int_{x^2}^5 e^{7t^2} \sqrt{5 + \cos(t)^3}\, dt.$$

Solution. *(i): It would be a mistake to try to compute this integral first! Rather, we can appeal directly to FTC (Theorem 3.16 (i)), which tells us that*

$$f'(x) = e^{7x^2} \sqrt{5 + \cos(x)^3}.$$

(ii): Again, we don't want to try to compute the integral. However, we can't appeal immediately to FTC since our interval of integration is $[x^2, 5]$*, rather than something of the form* $[a, x]$*. We first reverse the endpoints:*

$$g(x) = \int_{x^2}^5 e^{7t^2} \sqrt{5 + \cos(t)^3}\, dt = -\int_5^{x^2} e^{7t^2} \sqrt{5 + \cos(t)^3}\, dt.$$

Next, we note that what we get is a function of x^2*, namely*

$$g(x) = -f(x^2),$$

where f *is from part (i). We may therefore use the Chain Rule:*

$$g'(x) = -f'(x^2)\frac{d}{dx}x^2 = -e^{7(x^2)^2} \sqrt{5 + \cos(x^2)^3}\, 2x.$$

Example 3.19. *Evaluate the following*

(i) $$\int_0^8 2x - 7\sqrt[3]{x^4}\,dx.$$

(ii) $$\int_0^{\pi/2} 2\cos(t)\,dt.$$

(iii) $$\int_{-5}^5 5^x + x^5\,dx.$$

(iv) $$\int_1^6 x + \frac{1}{x}\,dx.$$

Solution. (i): We have

$$\int 2x - 7\sqrt[3]{x^4}\,dx = \int 2x - 7x^{4/3}\,dx = \frac{2}{2}x^2 - \frac{7}{7/3}x^{7/3} = x^2 - 3x^{7/3}.$$

Hence,

$$\begin{aligned}\int_0^8 2x - 7\sqrt[3]{x^4}\,dx &= x^2 - 3x^{7/3}\Big|_{x=0}^{8} \\ &= 8^2 - 3\cdot 8^{7/3} - (0^2 - 3\cdot 0^{7/3}) \\ &= 64 - 3\cdot 128 = -320.\end{aligned}$$

(ii): From now on, when the indefinite integral isn't too complicated, we won't do it separately.

$$\begin{aligned}\int_0^{\pi/2} 2\cos(t)\,dt &= 2\sin(t)|_{t=0}^{\pi/2} \\ &= 2\sin(\pi/2) - 2\sin(0) \\ &= 2\cdot 1 - 2\cdot 0 = 2.\end{aligned}$$

(iii): We have

$$\int_{-5}^{5} 5^x + x^5\,dx = \left.\frac{5^x}{\ln(5)} + \frac{x^6}{6}\right|_{x=-5}^{5}$$
$$= \frac{5^5}{\ln(5)} + \frac{5^6}{6} - \frac{5^{-5}}{\ln(5)} - \frac{(-5)^6}{6}$$
$$= \frac{1}{\ln(5)}\left(5^5 - \frac{1}{5^5}\right).$$

(iv): We have

$$\int_{1}^{6} x + \frac{1}{x}\,dx = \left.\frac{x^2}{2} + \ln(x)\right|_{x=1}^{6}$$
$$= \frac{6^2}{2} + \ln(6) - \frac{1^2}{2} - \ln(1)$$
$$= \frac{36}{2} + \ln(6) - \frac{1}{2} - 0 = \frac{35}{2} + \ln(6).$$

Example 3.20. *Compute*

$$\text{(i)} \quad \int_1^2 x^2 \sqrt{8 + x^3}\, dx.$$

$$\text{(ii)} \quad \int_0^{\sqrt{\pi}} t \cos\left(\tfrac{\pi}{3} - t^2\right)\, dt.$$

$$\text{(iii)} \quad \int_2^4 \frac{1 + t}{1 + t^2}\, dt.$$

Solution. (i): Here the indefinite integral is more complicated, so it is best to solve it first, then plug the answer into the FTC formula for the definite integral. To solve

$$\int x^2\sqrt{8+x^3}\,dx,$$

use the substitution

$$u = 8+x^3, \quad du = 3x^2\,dx.$$

Thus we have

$$\begin{aligned}\int x^2\sqrt{7+x^3}\,dx &= \int \sqrt{u}\,\frac{du}{3} \\ &= \frac{2}{9}u^{3/2}+C = \frac{2}{9}(8+x^3)^{3/2}+C.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_1^2 x^2\sqrt{8+x^3}\,dx &= \left.\frac{2}{9}(8+x^3)^{3/2}\right|_{x=1}^{2} \\ &= \frac{2}{9}16^{3/2}-\frac{2}{9}9^{3/2} = \frac{2}{9}(64-27) = \frac{74}{9}.\end{aligned}$$

It is crucial that we finished up the indefinite integration with an expression in terms of x and not the substituted variable u. It is not correct that

$$\int_1^2 x^2\sqrt{8+x^3}\,dx = \left.\frac{2}{9}u^{3/2}\right|_{u=1}^{2}.$$

(ii): Using the substitution

$$u = \tfrac{\pi}{3} - t^2, \quad du = -2t\,dt,$$

we have

$$\begin{aligned}\int t\cos\left(\tfrac{\pi}{3} - t^2\right)\,dt &= \int \cos(u)\,\frac{du}{-2}\\ &= -\frac{1}{2}\sin(u) + C\\ &= -\frac{1}{2}\sin(\tfrac{\pi}{3} - t^2) + C.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{\sqrt{\pi}} t\cos\left(\tfrac{\pi}{3} - t^2\right)\,dt &= -\frac{1}{2}\sin(\tfrac{\pi}{3} - t^2)\Big|_{t=0}^{\sqrt{\pi}}\\ &= -\frac{1}{2}(\sin(\tfrac{\pi}{3} - \pi) - \sin(\tfrac{\pi}{3}))\\ &= -\frac{1}{2}(\sin(-\tfrac{2\pi}{3}) - \sin(\tfrac{\pi}{3}))\\ &= -\frac{1}{2}(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2}.\end{aligned}$$

(iii): We split the indefinite integral into two parts, and use the substitition $u = 1 + t^2$ on the second part:

$$\begin{aligned}\int \frac{1+t}{1+t^2}\,dt &= \int \frac{1}{1+t^2}\,dt + \int \frac{t}{1+t^2}\,dt \\ &= \tan^{-1}(t) + \int \frac{1}{2u}\,du \\ &= \tan^{-1}(t) + \frac{\ln(u)}{2} + C \\ &= \tan^{-1}(t) + \frac{\ln(1+t^2)}{2} + C.\end{aligned}$$

Hence,

$$\begin{aligned}\int_2^4 \frac{1+t}{1+t^2}\,dt &= \left. \tan^{-1}(t) + \frac{\ln(1+t^2)}{2} \right|_{t=2}^{4} \\ &= \tan^{-1}(4) + \frac{\ln(17)}{2} - \tan^{-1}(2) - \frac{\ln(5)}{2}.\end{aligned}$$

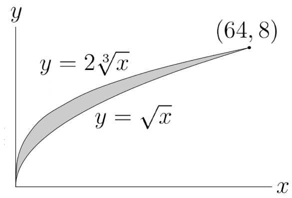
Example 3.21. *There is a bounded region enclosed by the curves*

$$y = 2\sqrt[3]{x}$$

and

$$y = \sqrt{x}.$$

Find its area. **Solution.** Let's first determine where these curves intersect.



They intersect when

$$2\sqrt[3]{x} = y = \sqrt{x},$$

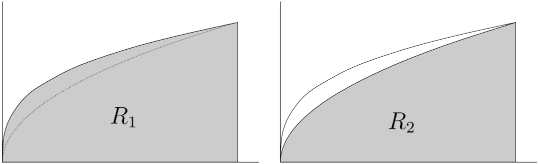
giving

$$64x^2 = x^3$$
$$\Rightarrow x^2(x - 64) = 0,$$

and the solutions are $(x, y) = (0, 0)$ and $(x, y) = (64, 8)$.

Thus far, we've learned how to find the area between a region bound by the x-axis and a curve. The area we need to find for this problem can be viewed as the difference of two areas that we already know. Namely, if R_1 is the region enclosed by $y = 2\sqrt[3]{x}$, $x = 0$, $y = 0$, and $x = 4$, and R_2 is the region enclosed by $y = \sqrt{x}$, $x = 0$, $y = 0$, and $x = 4$, then the area we need to find is

$$\text{Area}(R_1) - \text{Area}(R_2).$$



We now compute this,

$$\begin{aligned}
\text{Area}(R_1) - \text{Area}(R_2) &= \int_0^{64} 2\sqrt[3]{x}\,dx - \int_0^{64} \sqrt{x}\,dx \\
&= 2\tfrac{3}{4}x^{4/3} - \tfrac{2}{3}x^{3/2}\Big|_{x=0}^{64} \\
&= \tfrac{3}{2}(256 - 0) - \tfrac{2}{3}(512 - 0) \\
&= \frac{1152 - 1024}{3} = \frac{128}{3}.
\end{aligned}$$

More generally, for functions $f(x), g(x)$ with $f(x) \leq g(x)$, the area of a region enclosed by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is computed by the integral

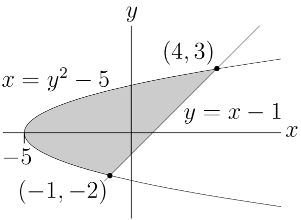
$$\int_a^b g(x) - f(x)\, dx.$$

Example 3.22. *Find the area of the region enclosed by the curves* $x = y^2 - 5$ *and* $y = x - 1$*.*

Solution. This problems differs a bit from previous ones, because the first curve is not in the form $y = f(x)$. Let's first compute where the curves intersect, by solving the two equations as a system. Substituting $y = x - 1$ into the first equation gives

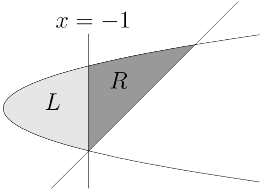
$$\begin{aligned} x &= (x-1)^2 - 5 \\ &= x^2 - 2x - 4, \\ 0 &= x^2 - 3x - 4 \\ &= (x-4)(x+1). \end{aligned}$$

The intersection points are therefore $(4,3)$ and $(-1,-2)$.



There are two ways that we can attack this problem.

(a): We divide the region by the line $x = -1$. On the left-hand side (L in the diagram below), we want to find the area between $y = -\sqrt{x+5}$ and $y = \sqrt{x+5}$ (from $x = -5$ to $x = -1$). On the right-hand side (R in the diagram below), we want to find the area between $y = x - 1$ and $y = \sqrt{x+5}$ (from $x = -1$ to $x = 4$).



Altogether, the area can be computed as

$$\begin{aligned}
\text{Area} &= \int_{-5}^{-1} \sqrt{x+5} - (-\sqrt{x+5})\,dx + \int_{-1}^{4} \sqrt{x+5} - (x-1)\,dx \\
&= 2 \cdot \frac{2}{3}(x+5)^{3/2}\Big|_{x=-5}^{-1} + \frac{2}{3}(x+5)^{3/2} - \frac{x^2}{2} + x\Big|_{x=-1}^{4} \\
&= \frac{4}{3}(8-0) + \frac{2}{3}(27-8) - \frac{1}{2}(16-1) + (4-(-1)) \\
&= \frac{125}{6}
\end{aligned}$$

(b): Looking at this sideways, we see that an easier approach is to swap the roles of the x and y coordinates.

Both curves can be put into the form $x = f(y)$; for the second one, it is $x = y + 1$.
The area is thus given by

$$\begin{aligned}
\text{Area} &= \int_{-2}^{3} y + 1 - (y^2 - 5)\, dy \\
&= -\frac{y^3}{3} + \frac{y^2}{2} + 6y \Big|_{y=-2}^{3} \\
&= -\frac{1}{3}(27 - (-8)) + \frac{1}{2}(9 - 4) + 6(3 - (-2)) \\
&= \frac{125}{6}.
\end{aligned}$$