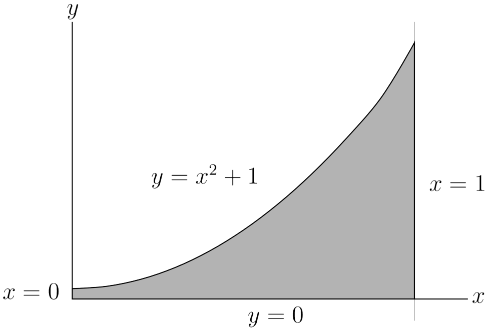
3-4. Area and the definite integral

The problem of computing area will tie into the main application of integrals.

Suppose we want to compute the area of some region.

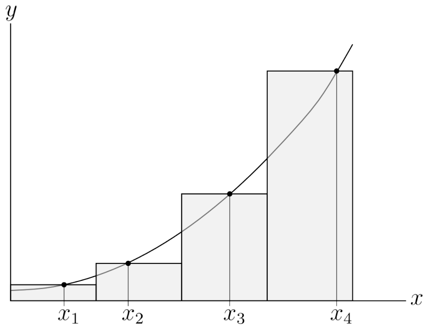
To begin, we need to define the area precisely; for now, say it is the area bound by the x-axis, some curve $y = f(x)$, and two vertical lines $x = a$ and $x = b$.

Let us use, as an example, the area bound by the x-axis, $y = x^2 + 1$, $x = 0$, and $x = 1$.



A first step towards (exactly) computing an area is finding a good way to estimate the error. We can partition the interval $[0, 1]$ into a number of subintervals; for now, let's partition it into the subintervals $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, 1]$. We can then pick points $x_1 \in [0, \frac{1}{4}]$, $x_2 \in [\frac{1}{4}, \frac{1}{2}]$, and so on.

Form the following rectangles:

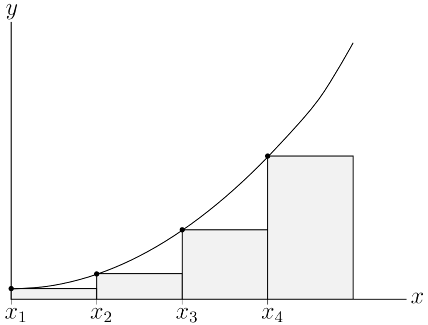


The total area of these rectangles (which is easy to compute!) gives us an approximation of the area we are interested in. Of course, the total area of these rectangles depends on the choice of points $x_1, \ldots, x_5$.

If we took left-hand endpoints, i.e., $x_i = \frac{i-1}{4}$, we get the estimate

$$\sum_{i=1}^{4}(\left(\frac{(i-1)}{4}\right)^2+1)\cdot\frac{1}{4} = \frac{1}{4}\left((0+1)+(\frac{1}{16}+1)+(\frac{1}{4}+1)+(\frac{9}{16}+1)\right) = \frac{39}{32} = 1.21875.$$

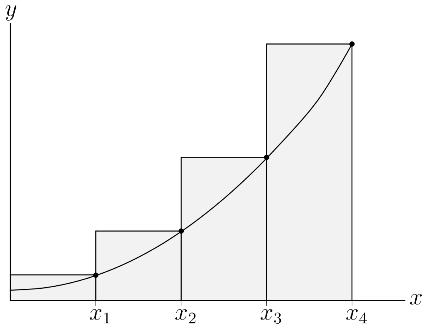
In this case, the rectangles are contained in the region we are interested in, so it is clear that this underestimates the correct area.



If we took right-hand endpoints, i.e., $x_i = \frac{i}{4}$, we get the estimate

$$\sum_{i=1}^{4}\left(\left(\tfrac{i}{4}\right)^2+1\right)\cdot\tfrac{1}{4} = \tfrac{1}{4}\left(\left(\tfrac{1}{16}+1\right)+\left(\tfrac{1}{4}+1\right)+\left(\tfrac{9}{16}+1\right)+(1+1)\right) = \tfrac{47}{32} = 1.46875.$$

Since these rectangles (together) completely contain the region, this overestimates the correct area.



Altogether, this tells us that

$$\text{Area} \in [\tfrac{39}{32}, \tfrac{47}{32}].$$

The average of these two estimates is $\frac{43}{32} = 1.34375$, which is $\frac{1}{8}$ away from each of the over- and the under-estimation. Hence, this average estimate is accurate to within $\frac{1}{8}$.

To get a certain improvement, we need to take a finer partition. If we started with the partition $[0, \frac{1}{10}]$, $[\frac{1}{10}, \frac{2}{10}]$, $\ldots$, then we arrive at the estimates

$$\frac{257}{200} = 1.285, \quad \frac{277}{200} = 1.385.$$

If we started with the partition $[0, \frac{1}{100}]$, $[\frac{1}{100}, \frac{2}{100}]$, $\ldots$, then we arrive at the estimates

$$\frac{26567}{20000} = 1.32835, \quad \frac{26767}{20000} = 1.33835.$$

In this situation, the two estimates (coming from using left- and right-hand endpoints of the subintervals respectively) are always under- and over-estimates. This is because the function in question $y = x^2 + 1$, is increasing on the given interval.

If we used a function that is decreasing, then using left-hand endpoints would instead give an overestimate, and right-hand endpoints would give an underestimate.

For a general function (which is neither increasing nor decreasing), we **do not know whether the estimates are greater or less than the correct area**.

We have a name for the sums appearing when we compute the areas of the rectangles: these are called Riemann sums. More precisely, let $f : [a, b] \to \mathbb{R}$ be a function, and partition $[a, b]$ into n equally sized subintervals,

$$I_1 = \left[a, a + \frac{b-a}{n}\right], \; I_2 = \left[a + \frac{b-a}{n}, a + 2\frac{b-a}{n}\right], \dots,$$
$$I_i = \left[\frac{n-i+1}{n}a + \frac{i-1}{n}b, \frac{n-i}{n}a + \frac{i}{n}b\right], \dots,$$
$$I_n = \left[b - \frac{b-a}{n}, b\right].$$

Pick a point $x_i \in I_i$ for each $i = 1, \dots, n$. Then the associated *Riemann sum* is

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ (the length of each interval in the partition).

Theorem 3.13. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. For each n, take a partition of $[a,b]$ into n equally sized subintervals $I_{1,n}, \ldots, I_{n,n}$ as above, and pick points $x_{i,n} \in I_{i,n}$. Then the Riemann sums converge to a limit, i.e.,*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}) \tfrac{1}{n}$$

exists. This limit does not depend on the choice of points $x_{i,n} \in I_{i,n}$.

We will not prove this theorem in this course (for the proof, take MA2509, "Analysis II").

A word of caution. the last statement of the above theorem tells us that the *limit* doesn't depend on the choices of points. This does not mean that the individual Riemann sums don't depend on the choices of points (and as we've seen in the example, the Riemann sums **do** depend on these choices).
Since the limit in the above theorem exists, and doesn't depend on the choices of points, it is a well-defined value, and we call it the **definite integral** of $f(x)$ on $[a,b]$. In the setting of the above theorem, we use the notation

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n})\,\tfrac{1}{n}.$$

Note that although our example of measuring area concerned a positive function, the above theorem and the definition of the definite integral does not require this assumption; we allow $f(x) < 0$ to occur.
We extend the definition of the definite integral to the case that $a > b$. Here we ask that $f(x)$ is a continuous function on $[b, a]$, and we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

A note about notation. When we write

$$\int_a^b f(x)\,dx,$$

the variable x is a bound, dummy variable. This expression has the exact same meaning as

$$\int_a^b f(t)\,dt,$$

(or with any other variable in place of t). It does **not** make sense to ask if $\int_a^b f(x)\,dx$ depends on x, or to have x outside the integrand, like

$$\int_a^b f(x)\,dx + x \quad \text{or} \quad \int_a^x f(x)\,dx.$$

(By contrast, the variable x does make sense outside of an **indefinite** integral: it does make sense to write $\int f(x)\,dx + x$.)

Example 3.14. *Let $f(x) = 5 - 3x$ on the interval $[0, 1]$. For each n, we partition $[0, 1]$ into n equally sized subintervals $I_{1,n}, \ldots, I_{n,n}$, so that*

$$I_{i,n} = [\tfrac{i-1}{n}, \tfrac{i}{n}].$$

Pick $x_{i,n} \in I_{i,n}$ as the right-hand endpoint, $x_i = \frac{i}{n}$. Then the Riemann sum is

$$\begin{aligned}\sum_{i=1}^{n} \left(5 - 3 \cdot \frac{i}{n}\right) \frac{1}{n} &= 5 \cdot \frac{n}{n} - \frac{3}{n^2} \sum_{i=1}^{n} i \\ &= 5 - \frac{3}{n^2} \cdot \frac{n(n+1)}{2} \\ &= 5 - \frac{3(n+1)}{2n}.\end{aligned}$$

Taking the limit, we get

$$\int_0^1 5 - 3x = \lim_{n \to \infty} 5 - \frac{3(n+1)}{2n} = 5 - \frac{3}{2} = \frac{7}{2}.$$

Proposition 3.15 (Basic properties of the definite integral). *Let $f(x), g(x)$ be continuous functions, let $K \in \mathbb{R}$, and let $a, b, c \in \mathbb{R}$. Then:*

(i) $\displaystyle\int_a^a f(x)\,dx = 0.$

(ii) $\displaystyle\int_a^b K f(x)\,dx = K\int_a^b f(x)\,dx.$

(iii) $\displaystyle\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

(iv) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

(v) $\displaystyle\int_a^b K'\,dx = K(b) - K(a).$

(vi) $\displaystyle\int_a^b f(x)\,dx \geq 0,$ *if* $f(x) \geq 0\ \forall x.$

(vii) $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x),$ *if* $f(x) \leq g(x)\ \forall x.$

(viii) $\displaystyle\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$

We will not prove these statements, although they are not too difficult (apart from (iv)), and make good exercises.