3-3. Substitution rule

In this section we learn the first technique for solving more complicated integrals.

Consider the integral

$$\int 12x^2 \sqrt[5]{4x^3+7}\,dx.$$

At first this one may look very difficult; but, there is a trick. Notice that if we let

$$u = 4x^3 + 7,$$

then we compute

$$du = 12x^2\,dx$$

(by differentiating with respect to x), and the integral simplifies

$$\int 12x^2 \sqrt[5]{4x^3+7}\,dx = \int \sqrt[5]{4x^3+7}\,(12x^2\,dx) = \int \sqrt[5]{u}\,du.$$

Is this legitimate? Yes, it is justified by the **chain rule** for differentiation.

Proposition 3.8 (Substitution rule). *Suppose that*

$$\int f(u)\,du = F(u) + C.$$

Then

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C.$$

Proof.

We compute, using the chain rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x) = f(g(x))\,g'(x).$$

Therefore,

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C.$$

□

We write the substitution rule succinctly as

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du, \quad \text{where } u = g(x).$$

The skill in using the substitution rule is to identify the right thing to substitute.

We want to pick some function $u = g(x)$ so that, after dividing by $g'(x)$, the integrand can be expressed purely in terms of u. If we find that there are some x's left over, we have probably chosen the wrong function $g(x)$.
To learn this skill, **you simply must practice!**

Example 3.9. *Evaluate the following integrals.*

(i) $$\int 12x^2 \sqrt[5]{4x^3+7}\,dx.$$

(ii) $$\int \left(1-\frac{2}{t}\right)\cos(t-2\ln(t))\,dt.$$

(iii) $$\int 7(6x-1)e^{3x^2-x}\,dx.$$

(iv) $$\int \cos(y)(1-5\sin(y))^7\,dy.$$

(v) $$\int \frac{t}{\sqrt[4]{1-2t^2}}\,dt.$$

(vi) $$\int x^5\,\frac{(2+\sqrt{1-x^6})^3}{\sqrt{1-x^6}}\,dx.$$

Solution.
(i): This is the example we already started. We use

$$u = 4x^3 + 7, \quad du = 12x^2\,dx,$$

so that

$$\begin{aligned}\int 12x^2 \sqrt[5]{4x^3+7}\,dx &= \int \sqrt[5]{4x^3+7}\,(12x^2\,dx)\\ &= \int \sqrt[5]{u}\,du\\ &= \frac{5}{6}u^{6/5} + C\\ &= \frac{5}{6}(4x^3+7)^{6/5} + C.\end{aligned}$$

Always remember to get rid of the substitution variable at the last step.
(I.e., express your final solution in terms of the original variable.)

(ii): Use

$$u = t - 2\ln(t), \quad du = 1 - \frac{2}{t}\,dt.$$

Then we have

$$\begin{aligned}\int \left(1 - \frac{2}{t}\right)\cos(t - 2\ln(t))\,dt &= \int \cos(u)\,du \\ &= \sin(u) + C \\ &= \sin(t - 2\ln(t)) + C.\end{aligned}$$

(iii): Use

$$u = 3x^2 - x, \quad du = 6x - 1\,dx.$$

This gives

$$\begin{aligned}\int 7(6x-1)e^{3x^2-x}\,dx &= \int 7e^u\,du \\ &= 7e^u + C \\ &= 7e^{3x^2-x} + C.\end{aligned}$$

(iv): Use

$$u = 1 - 5\sin(y), \quad du = -5\cos(y)\,dy$$

to get

$$\begin{aligned}
\int \cos(y)(1-5\sin(y))^7\,dy &= \int u^7\,\frac{du}{-5} \\
&= -\frac{1}{5\cdot 8}u^8 + C \\
&= -\frac{1}{40}(1-5\sin(y))^8 + C.
\end{aligned}$$

(v): Use

$$u = 1 - 2t^2, \quad du = -4t\,dt.$$

This gives

$$\begin{aligned}
\int \frac{t}{\sqrt[4]{1-2t^2}}\,dt &= \int -\frac{1}{4}u^{-1/4}\,du \\
&= -\frac{1}{4}\frac{4}{3}u^{3/4} + C \\
&= -\frac{1}{3}\sqrt[4]{(1-2t^2)^3} + C.
\end{aligned}$$

(vi): Use

$$u = 1 - x^6, \quad du = -6x^5\,du$$

and we get

$$\int x^5 \frac{(2+\sqrt{1-x^6})^3}{\sqrt{1-x^6}}\,dx = -\int \frac{(2+\sqrt{u})^3}{6\sqrt{u}}\,du.$$

To solve this, we need to do another substitution:

$$v = 2+\sqrt{u}, \quad dv = \frac{1}{2\sqrt{u}}\,du$$

to obtain

$$\begin{aligned} -\int \frac{(2+\sqrt{u})^3}{6\sqrt{u}}\,du &= -\int \frac{v^3}{3}\,dv \\ &= -\frac{1}{12}v^4 + C. \end{aligned}$$

Putting this together, we get

$$\begin{aligned} \int x^5 \frac{(2+\sqrt{1-x^6})^3}{\sqrt{1-x^6}}\,dx &= -\frac{1}{12}v^4 + C \\ &= -\frac{1}{12}(2+\sqrt{u})^4 + C \\ &= -\frac{1}{12}(2+\sqrt{1-x^6})^4 + C. \end{aligned}$$

After a long problem like this, it is always best to check your solution. Fortunately, this is easy to do, since it's just integration!
We have

$$\frac{d}{dx} - \frac{1}{12}(2+\sqrt{1-x^6})^4 = \frac{1}{12}4(2-\sqrt{1-x^6})^3\frac{1}{2\sqrt{1-x^6}}6x^5$$
$$= x^5\,\frac{2-\sqrt{1-x^6})^3}{\sqrt{1-x^6}},$$

which confirms that our answer is correct.
Note. If you are a clever clog, you might have seen how to do this problem with just one substitution,

$$w = 2+\sqrt{1-x^6}, \quad dw = \frac{1}{2\sqrt{1-x^6}}6x^5\,dx = \frac{x^5}{3\sqrt{1-x^6}}\,dx.$$

Example 3.10. *Evaluate the following.*

(i) $$\int \frac{7}{2x+3}\,dx.$$

(ii) $$\int \frac{7x}{2x^2+3}\,dx.$$

(iii) $$\int \frac{7x}{(2x^2+3)^2}\,dx.$$

(iv) $$\int \frac{7}{2x^2+3}\,dx.$$

Solution. (i): For this, use

$$u = 2x + 3, \quad du = 2\,dx$$

so that

$$\begin{aligned} \int \frac{7}{2x+3}\,dx &= \int \frac{7}{u}\,\frac{du}{2} \\ &= \frac{7}{2}\ln|u| + C \\ &= \frac{7}{2}\ln|2x+3| + C. \end{aligned}$$

(ii): Use

$$u = 2x^2 + 3, \quad du = 4x\,dx,$$

to get

$$\begin{aligned} \int \frac{7x}{2x^2+3}\,dx &= \int \frac{7}{4u}\,du \\ &= \frac{7}{4}\ln|u| + C \\ &= \frac{7}{4}\ln(2x^2+3) + C. \end{aligned}$$

(Note here, we dropped the absolute value sign because $2x^2 + 3$ is always positive. We don't have to do this – it would be acceptable to give the final answer as $\frac{7}{4}\ln|2x^2+3| + C$.)

(iii): Using the same substitution as in (ii), we obtain

$$\begin{aligned}\int \frac{7x}{(2x^2+3)^2}\,dx &= \int \frac{7}{4u^2}\,du \\ &= \int \frac{7}{4}u^{-2}\,du \\ &= \frac{7}{4}\cdot\frac{1}{-1}u^{-1}+C \\ &= -\frac{7}{4(2x^2+3)}+C.\end{aligned}$$

(iv): At first, it might look like we want to use the same substitution again, that is,

$$u = 2x^2 + 3, \quad du = 4x\,dx.$$

However, we don't have a term x, so we would get

$$\int \frac{7}{2x^2+3}\,dx = \int \frac{7}{4xu}\,du,$$

and this doesn't get rid of all occurrences of x, so we cannot proceed further. We **can't treat** x **as a constant**, since u is defined in terms of x, so we don't have

$$\int \frac{7}{4xu}\,du = \frac{7}{4x}\ln|u|.$$

We could solve for x in terms of u:

$$x = \pm\sqrt{\frac{u-3}{2}},$$

but the result is

$$\pm\int \frac{7}{4u\sqrt{(u-3)/2}}\,du,$$

which looks even more intimidating than the original integral. (Also we need to worry about the $\pm$, which is undesirable.)

Whenever we can't get rid of all occurrences of the old variable, it is a dead end; the substitution didn't work and we have to go back and try something else.
Here, we might instead recognise that the integrand

$$\frac{7}{2x^2 + 3}$$

looks something like the integrand

$$\frac{1}{x^2 + 1}$$

whose antiderivative is $\tan^{-1}(x)$.

We manipulate things to make it look more like the derivative of $\tan^{-1}(x)$:

$$\int \frac{7}{2x^2+3}\,dx = \frac{7}{3}\int \frac{1}{\frac{2}{3}x^2+1}\,dx = \frac{7}{3}\int \frac{1}{(\sqrt{2/3}x)^2+1}\,dx$$

and now we see that the correct substitution is

$$u = \sqrt{\frac{2}{3}}x, \quad du = \sqrt{\frac{2}{3}}\,dx.$$

Thus,

$$\begin{aligned}
\int \frac{7}{2x^2+3}\,dx &= \frac{7}{3}\int \frac{1}{(\sqrt{2/3}x)^2+1}\,dx \\
&= \frac{7}{3}\int \frac{1}{u^2+1}\sqrt{\frac{3}{2}}\,du \\
&= \frac{7\sqrt{3}}{3\sqrt{2}}\tan^{-1}(u)+C \\
&= \frac{7\sqrt{3}}{3\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{2}{3}}x\right)+C.
\end{aligned}$$

Example 3.11. *Solve:*

$$\int \frac{t+1}{\sqrt{1-9t^2}}\,dt.$$

Solution. Whenever the integrand is a sum of two things, it is advisable to break it into the two parts using linearity:

$$\int \frac{t+1}{\sqrt{1-9t^2}}\,dt = \int \frac{1}{\sqrt{1-9t^2}}\,dt + \int \frac{t}{\sqrt{1-9t^2}}\,dt.$$

Now, we need to solve the two integrals separately:

$$\text{(i)} \int \frac{t}{\sqrt{1-9t^2}}\,dt \quad \text{and}$$

$$\text{(ii)} \int \frac{1}{\sqrt{1-9t^2}}\,dt.$$

(i) We do this by the substitution

$$u = 1 - 9t^2, \quad du = -18t\,dt.$$

This gives

$$\begin{aligned}\int \frac{t}{\sqrt{1-18t^2}}\,dt &= \int \frac{1}{-18\sqrt{u}}\,du \\ &= -\frac{2}{18}\sqrt{u} + C' \\ &= -\frac{1}{9}\sqrt{1-t^2} + C',\end{aligned}$$

C' any constant.

(ii): While it may seem like we want to again substitute

$$u = 1 - 9t^2, \quad du = -18t\, dt,$$

there isn't a "t" available for the du part. Instead, we recognise that the integrand looks similar to

$$\frac{1}{\sqrt{1-t^2}},$$

whose antiderivative is

$$\sin^{-1}(t).$$

We thus do a substitution to arrive at this exactly:

$$u = 3t, \quad du = 3\, dt.$$

This leads to

$$\begin{aligned} \int \frac{1}{\sqrt{1-9t^2}}\, dt &= \int \frac{1}{3\sqrt{1-u^2}}\, du \\ &= \frac{1}{3}\sin^{-1}(u) + C'' \\ &= \frac{1}{3}\sin^{-1}(3t) + C''. \end{aligned}$$

C'' any constant.

Putting this together, we get

$$\int \frac{t+1}{\sqrt{1-9t^2}}\,dt = -\frac{1}{9}\sqrt{1-9t^2} + \frac{1}{3}\sin^{-1}(3t) + C.$$

(We combined the two integration constants into one. It is best practice to use different symbols for different integration constants, but it probably won't cause much confusion if this isn't done.) □

In the previous two examples, we saw that sometimes similar-looking integrands may require very different methods.

As mentioned before, we simply **cannot** solve every integral problem.

One way to say this is that we just don't have names for all the functions we would need. However, we might make up new functions, and then try to solve other integrals in terms of these.

For example, it is a fact that there is a differentiable function $G : \mathbb{R} \to \mathbb{R}$ which satisfies

$$G'(t) = \frac{\sin(t)}{t}$$

for all $t \neq 0$; however, this function cannot be expressed in terms of elementary functions (polynomials, trig, exponential, logarithm).

Example 3.12. *Using the function G above to express the answer, solve*

$$\int \sin(e^t)\,dt.$$

Solution. Use the substitution

$$u = e^t, \quad du = e^t\,dt.$$

This gives

$$\begin{aligned}\int \sin(e^t)\,dt &= \int \frac{\sin(e^t)}{e^t}\,e^t dt \\ &= \int \frac{\sin(u)}{u}\,du \\ &= G(u) + C \\ &= G(e^t).\end{aligned}$$

□