3-2. Computing integrals: the basics

It is very easy to differentiate – there are rules that work every time.

Antidifferentiation (finding antiderivatives) is not easy; it is not always possible.

For example,

$$f(x) = e^{-x^2}$$

doesn't have a closed form antiderivative (we can't write down a formula for its antiderivative, except by making up new functions).

Even when it is possible to find an antiderivative, **there are no rules that always work**.

Instead, we shall learn a number of techniques for finding antiderivatives.

For any integration problem, we may need to try a few approaches, and often combine techniques, before discovering the solution.

We start with some integrals we just know by differentiating basic functions.

For polynomials:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

In fact, this works for any value of n (not just integers) **except** $n = -1$.

Recalling the derivatives of trigonometric functions, we have

$$\begin{aligned}
\int \cos(x)\,dx &= \sin(x) + C,\\
\int \sin(x)\,dx &= -\cos(x) + C,\\
\int \sec(x)^2\,dx &= \tan(x) + C,\\
\int \sec(x)\tan(x)\,dx &= \sec(x) + C,\\
\int \csc(x)^2\,dx &= -\cot(x) + C,\\
\int \csc(x)\cot(x)\,dx &= -\csc(x) + C,\\
\int \frac{1}{x^2+1}\,dx &= \tan^{-1}(x) + C,\\
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1}(x) + C.
\end{aligned}$$

Recalling the derivatives of exponential functions, we have

$$\int e^x\,dx = e^x + C,$$
$$\int a^x\,dx = \frac{1}{\ln(a)}a^x + C,$$
$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C.$$

(Note that, we don't have a formula for the antiderivative of $\ln(x)$ – this will require some later techniques.)

Example 3.6. *Evaluate the following:*

$$\text{(i)} \quad \int 7t^3 - 2t^{-4}\, dt$$

$$\text{(ii)} \quad \int 5\sqrt{x} - \frac{2}{\sqrt[3]{x^2}}\, dx$$

$$\text{(iii)} \quad \int \frac{4u^3 + 6}{u^4}\, du$$

$$\text{(iv)} \quad \int \sin(2x) + 4e^x\, dx$$

$$\text{(v)} \quad \int \cos(x+1) + \frac{5}{1+x^2} - \sqrt{x^7}\, dx$$

Solution.

(i):

$$\int 7t^3 - 2t^{-4}\,dt = \frac{7}{4}t^4 - \frac{2}{-3}t^{-3} + C$$
$$= \frac{7}{4}t^4 + \frac{2}{3t^3} + C.$$

(ii):

$$\int 5\sqrt{x} - \frac{2}{\sqrt[3]{x^2}}\,dx = \int 5x^{1/2} - 2x^{-2/3}\,dx$$
$$= \frac{5}{3/2}x^{3/2} - \frac{2}{1/3}x^{1/3} + C$$
$$= \frac{10}{3}x^{3/2} - 6x^{1/3} + C.$$

(iii):

$$\int \frac{4u^3 + 6}{u^4}\,du = \int 4u^{-1} + 6u^{-4}\,du$$
$$= 4\ln|u| + \frac{6}{-3}u^{-3} + C$$
$$= 4\ln|u| - 2u^{-3} + C.$$

(iv):

$$\int \sin(2x) + 4e^x \, dx = -\frac{1}{2}\cos(2x) + 4e^x + C.$$

Here we didn't use the integrals derived above, but instead we remember that the derivative of $\cos(2x)$ is $-2\sin(2x)$.

(v):

$$\int \cos(x+1) + \frac{5}{1+x^2} - \sqrt{x^7} \, dx = \sin(x+1) + 5\tan^{-1}(x) - \frac{2}{9}x^{9/2} + C.$$

Again, we compute that the $\frac{d}{dx}\sin(x+1) = \cos(x+1)$ in order to deal with the first term.

Example 3.7. *Solve*

$$\int \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx.$$

Solution. Although our later techniques will give us other options for dealing with this one, we can do this one by simplifying the integrand. Recall the double-angle formula

$$\sin(2t) = 2\sin(t)\cos(t),$$

and thus,

$$\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2}\sin(x).$$

Therefore,

$$\begin{aligned}\int \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx &= \frac{1}{2}\int \sin(x)\,dx \\ &= -\frac{1}{2}\cos(x) + C.\end{aligned}$$

Even when we have other tools at our disposal, remember as in the previous example, to **try to simplify the integrand** as one possible way to solve the integral.