3. Integration

Example 3.1. *What function has, as a derivative, the function*

$$f(x) = x^5 + 3x - 1?$$

Solution. The question asks to find $F(x)$ such that $F'(x) = f(x)$.
As a warm-up, take the derivative of $f(x)$. It is

$$f'(x) = 5x^4 + 3.$$

This doesn't actually help, but reminds us how to differentiate.

Differentiating a power of x, say x^n, is done by multiplying by the power and lowering the power by 1, i.e.,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

To go backwards, we must therefore divide by the power plus one and then raise the power by one. So, we can take F to be

$$F(x) = \frac{1}{6}x^6 + \frac{3}{2}x^2 - x.$$

Note that this isn't the only solution, for example we could instead take

$$F(x) = \frac{1}{6}x^6 + \frac{3}{2}x^2 - x + 2,$$

because the derivative of a constant (2 in this case) is 0.

Proposition 3.2. *Suppose that $f(x)$ is a function and $F(x)$, $G(x)$ are functions which satisfy*

$$F'(x) = f(x), \quad \textit{and} \quad G'(x) = f(x).$$

Then $F(x) - G(x)$ is a constant.

Proof.

Set $H(x) = F(x) - G(x)$. Then

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0.$$

We must show that $H(x)$ is a constant, i.e., that $H(x) = H(y)$ for all x, y.

For this, suppose that $x < y$. By the Mean Value Theorem, there exists $z \in (x, y)$ such that

$$H(y) - H(x) = H'(z)(y - x).$$

Since $H'(z) = 0$, it follows that $H(x) = H(y)$. □

Definition 3.3. *If $f(x)$ is a function, an antiderivative of $f(x)$ is any function $F(x)$ such that*

$$F'(x) = f(x)$$

as functions. In this case, we write

$$\int f(x)\,dx = F(x) + C, \quad C \text{ is any constant.}$$

This is called the **indefinite integral** *of $f(x)$;*

- *$f(x)$ is the* **integrand**,
- *x is the* **variable of integration**,
- *and C is the* **constant of integration**.

(By the previous proposition, this is the general form of an antiderivative of $f(x)$.)

Example 3.4. *Evaluate the indefinite integral*

$$\int x^3 + 3x\,dx.$$

Solution. *First we find a function* $F(x)$ *such that*

$$F'(x) = x^3 + 3x.$$

As in the previous example, we can see that the function

$$F(x) = \frac{1}{4}x^4 + \frac{3}{2}x^2$$

works. Thus, the indefinite integral is

$$\int x^3 + 3x\,dx = \frac{1}{4}x^4 + \frac{3}{2}x^2 + C, \quad C \text{ any constant.}$$

Note: **it is important to remember to write dx at the end of the integral**, for two reasons:

(i) The dx tells us where the integrand stops. If you don't write it, the meaning of what you've written is ambiguous. For example, if we wrote

$$\int x^3 - 3x^2 + 5,$$

it isn't clear if we mean

$$\begin{aligned} \int x^3 - 3x^2 + 5\,dx &= \frac{1}{4}x^4 - x^3 + 5x + C, \\ \int x^3 - 3x^2\,dx + 5 &= \frac{1}{4}x^4 - x^3 + C + 5, \quad \text{or} \\ \int x^3\,dx - 3x^2 + 5 &= \frac{1}{4}x^4 + C - 3x^2 + 5. \end{aligned}$$

(ii) The dx tells us the variable of integration (in this case, x). We are allowed to do integration with different variables of integration (and this will soon be important). While

$$\int 3x^2\,dx = x^3 + C,$$

we likewise have

$$\int 3t^2\,dt = t^3 + C.$$

In some cases, we might have multiple variables, but only one can be the variable of integration – the others are treated as constants. For example,

$$\int Kx^2\,dx = \frac{K}{3}x^3 + C,$$

whereas if we wrote $\int Kx^2\,dK$, we would have to treat x as constant, so the answer is

$$\int Kx^2\,dK = \frac{x^2}{2}K^2 + C.$$

Similarly,

$$\int x^n \, dx = \frac{1}{(n+1)} x^{n+1} + C,$$

while

$$\int x^n \, dn = \frac{x^n}{\ln(x)} + C.$$

(We usually try to avoid using n as the variable of integration, because usually n stands for an integer variable.)

Proposition 3.5 (Linearity of the integral).

(i)

$$\int K f(x)\, dx = K \int f(x)\, dx,$$

where K *is any scalar (constant). This includes the case of* $K < 0$, *for example*

$$\int -f(x)\, dx = -\int f(x)\, dx.$$

(ii)

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx.$$

Proof.

These both follow from linearity of differentiation: say

$$\int f(x)\,dx = F(x) + C,$$

which means that $F'(x) = f(x)$. Then:

(i) We know that $(KF)'(x) = KF'(x) = Kf(x)$, so that

$$\int Kf(x)\,dx = KF(x) + D, \quad D \text{ any constant.}$$

(ii) Suppose likewise that $G'(x) = g(x)$, so that

$$\int g(x)\,dx = G(x) + E, \quad E \text{ any constant.}$$

Then $(F+G)'(x) = F'(x) + G'(x) = f(x) + g(x)$. Thus, (avoiding writing the constants of integration), we have

$$\int f(x) + g(x)\,dx = F(x) + G(x) = \int f(x)\,dx + \int g(x)\,dx.$$

□

By contrast, note that

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx$$

and

$$\int \frac{f(x)}{g(x)}\,dx \neq \int f(x)\,dx \bigg/ \int g(x)\,dx.$$

(Because analogous formulae don't hold for differentiation.)

This is one of the things that makes integration interesting!