UNIVERSITY OF ABERDEEN **SESSION 2016–17**

Degree Examination SOLUTIONS

MA1005 Calculus I

December 13th 2016 (09:00 - 11:00)

SECTION A – answer all questions

1. **(a)** **(2 marks.)** $\lim_{x\to 1} \frac{x^2+4x-5}{x^2+3x-4} = \lim_{x\to 1} \frac{(x+5)(x-1)}{(x+4)(x-1)} = \lim_{x\to 1} \frac{(x+5)}{(x+4)} = \frac{(1+5)}{(1+4)} = \frac{6}{5}$.

(b) **(2 marks.)** $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1} = \lim_{x\to 1} \frac{(x-1)^2}{(x+1)(x-1)} = \lim_{x\to 1} \frac{(x-1)}{(x+1)} = \frac{0}{2} = 0$.

(c) **(2 marks.)** $\lim_{t\to 0} \frac{\sqrt{3+t}-\sqrt{3-t}}{t} = \lim_{t\to 0} \frac{(\sqrt{3+t}-\sqrt{3-t})(\sqrt{3+t}+\sqrt{3-t})}{t(\sqrt{3+t}+\sqrt{3-t})} = \lim_{t\to 0} \frac{(3+t)-(3-t))}{t(\sqrt{3+t}+\sqrt{3-t})} = \lim_{t\to 0} \frac{2t}{t(\sqrt{3+t}+\sqrt{3-t})} = \lim_{t\to 0} \frac{2}{(\sqrt{3+t}+\sqrt{3-t})} = \frac{2}{(\sqrt{3}+\sqrt{3})} = \frac{1}{\sqrt{3}}$

(d) **(2 marks.)** $\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{5}{x^2+x-6}\right] = \lim_{x\to 2} \left[\frac{1}{x-2} - \frac{5}{(x-2)(x+3)}\right] = \lim_{x\to 2} \frac{(x+3)-5}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+3)} = \lim_{x\to 2} \frac{1}{x+3} = \frac{1}{5}$

(e) **(2 marks.)** $\lim_{\theta\to 0} \frac{\theta-\theta^2}{\sin(\theta)} = \lim_{\theta\to 0} \frac{1-\theta}{\sin(\theta)/\theta} = \frac{\lim_{\theta\to 0} 1-\theta}{\lim_{\theta\to 0} \sin(\theta)/\theta} = \frac{1}{1} = 1$

2. **(a) (2 marks.)** $f(x) = \frac{x^3}{x^2+1}$, so $f'(x) = \frac{3x^2(x^2+1)-x^3(2x)}{(x^2+1)^2} = \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$

(2 marks.) $g(x) = x^2\sin^2(x)$, so $g'(x) = 2x\sin^2(x)+x^2(2\sin(x)\cos(x)) = 2x\sin(x)[\sin(x)+x\cos(x)]$.

(2 marks.) $h(x) = \frac{1}{\sqrt{3\sin(x)-1}} = (3\sin(x)-1)^{-1/2}$, so that $h'(x) = -\frac{1}{2}(3\sin(x)-1)^{-3/2}(3\cos(x)) = -\frac{3}{2}\cos(x)(3\sin(x)-1)^{-3/2}$

(2 marks.) $k(t) = \sqrt[3]{t^4} + \sqrt[4]{t^3} = (t^4)^{1/3} + (t^3)^{1/4} = t^{4/3} + t^{3/4}$ so that $k'(t) = \frac{4}{3}t^{1/3} + \frac{3}{4}t^{-1/4}$

(b) (2 marks.) $\sin(y)+xy^2 = 4$, so differentiating both sides gives $y'\cos(y)+y^2+2xyy' = 0$, so $y'(\cos(y)+2xy)+y^2 = 0$, so finally $y' = \frac{-y^2}{\cos(y)+2xy}$.

3. (a) (5 marks.)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x^2+1)-((x+h)^2+1)}{((x+h)^2+1)(x^2+1)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2+1-x^2-2xh-h^2-1}{((x+h)^2+1)(x^2+1)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2xh-h^2}{((x+h)^2+1)(x^2+1)}\right] \\
&= \lim_{h\to 0} \frac{-2x-h}{((x+h)^2+1)(x^2+1)} \\
&= \frac{-2x-0}{((x+0)^2+1)(x^2+1)} \\
&= \frac{-2x}{(x^2+1)^2}
\end{aligned}$$

(b) (5 marks.) Let $\epsilon > 0$ and define $\delta = \min(1, \epsilon/7)$. Suppose that $0 < |x-2| < \delta$.

Since $\delta \leqslant 1$ and $|x-2| < \delta$, we have $|x-2| < 1$, or in other words $-1 < x-2 < 1$, so that $5 < x+4 < 7$, and consequently $|x+4| < 7$.

Now

$$|(x^2+2x+3)-11| = |x^2+2x-8| = |(x+4)(x-2)| = |x+4|\cdot|x-2| < 7\delta \leqslant 7\epsilon/7 = \epsilon$$

so that $|(x^2+2x+3)-11| < \epsilon$ as required.

4. **(a) (2 marks.)**

$$\begin{aligned} f'(x) &= \frac{1(x^2+1) - x(2x+0)}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2} \end{aligned}$$

(4 marks.)

$$\begin{aligned} f''(x) &= \frac{(0-2x)(x^2+1)^2 - (1-x^2)\cdot 2(x^2+1)\cdot(2x+0)}{(x^2+1)^4} \\ &= \frac{(0-2x)(x^2+1) - (1-x^2)\cdot 2\cdot(2x+0)}{(x^2+1)^3} \\ &= \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3} \\ &= \frac{2x^3 - 6x}{(x^2+1)^3} \end{aligned}$$

(b) (3 marks.) The critical numbers are those x for which $f'(x)$ is not defined, of which there are none, and those x for which $f'(x) = 0$, i.e. $x^2 - 1 = 0$, i.e. $x = 1$ and $x = -1$.

For $x = 1$ we have $f''(x) = -4/8 < 0$ so that by the second derivative test there is a local maximum at this point.

For $x = -1$ we have $f''(x) = 4/8 > 0$ so that by the second derivative test there is a local minimum at this point.

(c) (3 marks.) The derivative $f'(x)$ has the same sign as $1 - x^2 = (1+x)(1-x)$. This vanishes for $x = 1, -1$ as above.

For $x < -1$ we see that $(1+x)$ is negative and $1-x$ is positive, so that $f'(x)$ is negative. Thus f is decreasing on $(-\infty, -1)$.

For $-1 < x < 1$ we see that $(1+x)$ is positive and $(1-x)$ is positive, so that $f'(x)$ is positive. Thus f is increasing on $(-1, 1)$.

For $1 < x$ we see that $(1+x)$ is positive and $(1-x)$ is negative, so that $f'(x)$ is negative. Thus f is decreasing on $(1, \infty)$.

(d) (4 marks.) The function f is rational and its domain is $(-\infty, \infty)$, so that f is continuous on the closed interval $[1/2, 3]$. It follows that we may apply the closed interval method. There is only one critical point in the interval, $x = 1$, and $f(1) = 1/2$. And at the endpoints of the interval we have $f(1/2) = 2/5$ and $f(3) = 3/10$.

So there is an absolute maximum of $1/2$ at $x = 1$ and an absolute minimum of $3/10$ at $x = 3$.

(e) (4 marks.) The function f is a rational function, and its domain is $(-\infty, \infty)$. So in particular it is continuous on $[-1, 1]$. Now $f(-1) = -1/2$ and $f(1) = 1/2$, so that $f(-1) < 1/3 < f(1)$. The intermediate value theorem therefore applies, and shows that there is $c \in (-1, 1)$ for which $f(c) = 1/3$.

SECTION B – answer two questions

5. **(a) (5 marks.)** We have $s'(x) = p'(q(x^2)) \cdot q'(x^2) \cdot 2x$ so that $s'(2) = p'(q(4)) \cdot q'(4) \cdot 4 = p'(2) \cdot 5 \cdot 4 = 3 \cdot 5 \cdot 4 = 60$.

And $t'(x) = \frac{1 \cdot q(x) - x \cdot q'(x)}{q(x)^2}$ so that $t'(4) = \frac{q(4) - 4q'(4)}{q(4)^2} = \frac{2 - 4 \cdot 5}{2^2} = \frac{-18}{4} = -9/2$.

(b) (5 marks.) By definition

$$k'(0) = \lim_{h \to 0} \frac{k(0+h) - k(0)}{h} = \lim_{h \to 0} \frac{h \cdot |h| \cdot \sin(1/h) - 0}{h} = \lim_{h \to 0} |h| \cdot \sin(1/h).$$

We will use the squeeze theorem to show that this limit is 0. Define p, q and r as follows.

$$p(h) = -|h|, \qquad q(h) = |h| \sin(1/h), \qquad r(h) = |h|.$$

Then since $-1 \leqslant \sin(1/h) \leqslant 1$, we have $-|h| \leqslant |h| \sin(1/h) \leqslant |h|$, or in other words $p(h) \leqslant q(h) \leqslant r(h)$. Also, observe that $\lim_{h \to 0} p(h) = 0 = \lim_{h \to 0} r(h)$. Then the squeeze theorem shows that $\lim_{h \to 0} q(h) = 0$. In other words $k'(0) = 0$ as required.

6. **(a) (5 marks.)** Let $\epsilon > 0$ and define $\delta = \min(1, 44\epsilon)$. Suppose that $0 < |x-4| < \delta$.

Since $\delta \leqslant 1$ and $|x-4| < \delta$, we have $|x-4| < 1$, so that $-1 < x-4 < 1$, so that $11 < x+8 < 13$, so that $11 < |x+8|$ and so $\frac{1}{|x+8|} < \frac{1}{11}$.

Then

$$\left|\frac{x+5}{x+8} - \frac{3}{4}\right| = \left|\frac{4(x+5)-3(x+8)}{4(x+8)}\right| = \left|\frac{x-4}{4(x+8)}\right| = \frac{1}{4}\frac{|x-4|}{|x+8|} < \frac{1}{4}\cdot\delta\cdot\frac{1}{11} = \frac{1}{44}\cdot\delta \leqslant \frac{1}{44}\cdot 44\epsilon = \epsilon$$

so that $\left|\frac{x+5}{x+8} - \frac{3}{4}\right| < \epsilon$ as required.

(b) (5 marks.) For the function k to be even we need $k(-x) = k(x)$, or in other words

$$x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = x^6 - ax^5 + bx^4 - cx^3 + dx^2 - ex + f$$

so that $a = -a$, $b = b$, $c = -c$, $d = d$, $e = -e$, and consequently $a = c = e = 0$. Thus

$$k(x) = x^6 + bx^4 + dx^2 + f.$$

Then

$$k'(x) = 6x^5 + 4bx^3 + 2dx = 2x(3x^4 + 2bx^2 + d).$$

Since $k'(x)$ exists for all x, the critical points of k are just the x for which $k'(x) = 0$. Observe that $k'(0) = 0$ automatically. To have $k'(1) = 0$ we need $3 + 2b + d = 0$, and the condition $k'(-1) = 0$ is the same. To have $k'(2) = 0$ we need $48 + 8b + d = 0$, and the condition $k'(-2) = 0$ is the same. Solving, we find that $b = -15/2$ and $d = 12$. For $k(0) = 1$ we need $f = 1$.

So the answer is $a = c = e = 0$, $b = -15/2$, $d = 12$, $f = 1$.

7. **(a) (6 marks.)** For f to be continuous at a means that $\lim_{x\to a} f(x) = f(a)$, or in other words that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$, or in other words that $\lim_{x\to a^+}(x+2) = \lim_{x\to a^-}(4x^2+x+1) = 4a^2+a+1$, or in other words $a+2 = 4a^2+a+1$. This is equivalent to $4a^2-1=0$, or in other words $a = \pm 1/2$. Since the question asks for positive a, we have $a = 1/2$.

We will now show that $f'(1/2)$ does not exist by showing that $\lim_{h\to 0^+} \frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h}$ and $\lim_{h\to 0^-} \frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h}$ are not equal. First,

$$\lim_{h\to 0^+} \frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h} = \lim_{h\to 0^+} \frac{(\frac{1}{2}+h)+2-\frac{5}{2}}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Next,

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h} &= \lim_{h\to 0^-} \frac{4(\frac{1}{2}+h)^2+(\frac{1}{2}+h)+1-\frac{5}{2}}{h} \\
&= \lim_{h\to 0^-} \frac{1+4h+4h^2+\frac{1}{2}+h+1-\frac{5}{2}}{h} \\
&= \lim_{h\to 0^-} \frac{5h+4h^2}{h} \\
&= \lim_{h\to 0^-} 5+4h \\
&= 5.
\end{aligned}$$

Since the left and right limits are not equal, this completes the proof.

(b) (4 marks.) An example of such a function would be the graph that is constant on each interval of the form $[2n, 2n+1]$ and that has slope 1 on each interval of the form $[2n+1, 2n+2]$, and that is continuous.

UNIVERSITY OF ABERDEEN **SESSION 2016–17**

Degree Examination SETTER'S COMMENTS

MA1005 Calculus I

December 13th 2016 (09:00 - 11:00)

---

0. This is a 2-hour exam.

The exam counts for 70 percent of a student's grade on the course. The other 30 percent come from two in-class tests and one assessed homework.

The questions are all marked out of 10, except for question 4 which is marked out of 20, for a total mark of 70. The questions are almost all of a standard kind that the students will be familiar with from examples sheets, class tests and homeworks. Students who have studied the tutorial sheets and past papers should not be surprised by anything. All questions are unseen.

Section A consists of standard questions. A student who understands the basics of the course should be able to get a passing mark here with little difficulty.

Section B consists of questions with two parts, both relatively hard, the latter especially so. These are intended to distinguish the good students from the best.

The class will be informed of the exam format, and of the intended difficulty of the Section B questions, well in advance.

Please note: Last year's exam had a rather high failure rate, but a fine number of first-class scores. So this year I have tried to make the easier questions easier (questions 1 and 2 especially so, and question 4 somewhat), and the hard questions a little harder.

1. These are routine problems: the trick in the second part is well known, as is the special limit in the final part. This has deliberately been made easier than last year in order to allow the weaker students to collect some marks.

2. This question is routine. It has deliberately been made easier than last year in order to allow the weaker students to collect marks.

3. These questions are both routine, and the question is at roughly the same difficulty as in previous exams.

4. This is extremely similar to a question from last year, but slightly easier, and with (hopefully) more precise questions.

5. The first part is a familiar kind of question (that students have found hard in the past). The second part is hard and requires some joined-up thinking on the part of the students.

6. The first part is routine but relatively hard. The second part is similar to hard questions in the past.

7. Problems similar to these have appeared in previous years. However, piecewise formulas have been asked for in the past rather than sketches.

UNIVERSITY OF ABERDEEN **SESSION 2016–17**

DEGREE EXAMINATION
MA1005 Calculus I
December 13th 2016 (09:00 - 11:00)

Cover Page

Paper Setter	**Dr.R Hepworth**
Paper Vetter	
External	

Please note below any Tables or other documents to be provided during the exam.

Before paper goes to External Examiner.....

Please make sure that you have provided *please tick boxes*

- Solutions, with provisional marking scheme ☐
- Setter's comments (and any comments that the vetter wishes to include) ☐
- CA information to date (if appropriate) ☐

Declaration: Final Version of Paper checked by Setter. Please Initial: ☐

After return from External Examiner.....

Please make sure that you have provided

- Complete information on CA (if appropriate) ☐
- Response to External's comments ☐

Declaration: Revised Version checked by Setter. Please Initial: ☐