University of Aberdeen

Advanced Mathematics I-1
Answers and Worked Solutions

1. **(a)** -1.25.

(b) f has domain $[-2,4]$ and range $[-2,3]$.

(c) g has domain $[-3,4]$ and range $[-0.25,4]$. (Here -0.25 is an estimate.)

2. **(a)** $\{x \mid x \neq -2, 2\}$, or in other words $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$.

(b) $\mathbb{R}$.

(c) $\{x \mid x \neq 1, -2\}$, or in other words $(-\infty,-2) \cup (-2,1) \cup (1,\infty)$

(d) *Worked Solution:* $\sqrt{2-t}$ is defined when $2-t \geqslant 0$, or in other words when $t \leqslant 2$. $\sqrt{2+t}$ is defined when $2+t \geqslant 0$, or in other words when $t \geqslant 2$. So $\sqrt{2-t}+\sqrt{2+t}$ is defined when both condition s$t \geqslant -2$ and $t \leqslant 2$ are true. Thus the range is $[-2,2]$.

(e) $(-\infty,0) \cup (6,\infty)$.

(f) *Worked Solution:* $\dfrac{1}{\sqrt[3]{u^2-6u}}$ is defined when $\sqrt[3]{u^2-6u}$ is defined and nonzero. Now $\sqrt[3]{u^2-6u}$ is always defined, and it is nonzero when u^2-6u is nonzero, or in other words when $u \neq 0, 6$. Thus the domain is $\{u \mid u \neq 0, 6\}$, or in other words $(-\infty,0) \cup (0,6) \cup (6,\infty)$.

(g) *Worked Solution:* For the function to be defined we require that both square roots exist, or in other words that

$$1+t \geqslant 0$$

for $\sqrt{1+t}$ to exist, and that

$$4-\sqrt{1+t} \geqslant 0$$

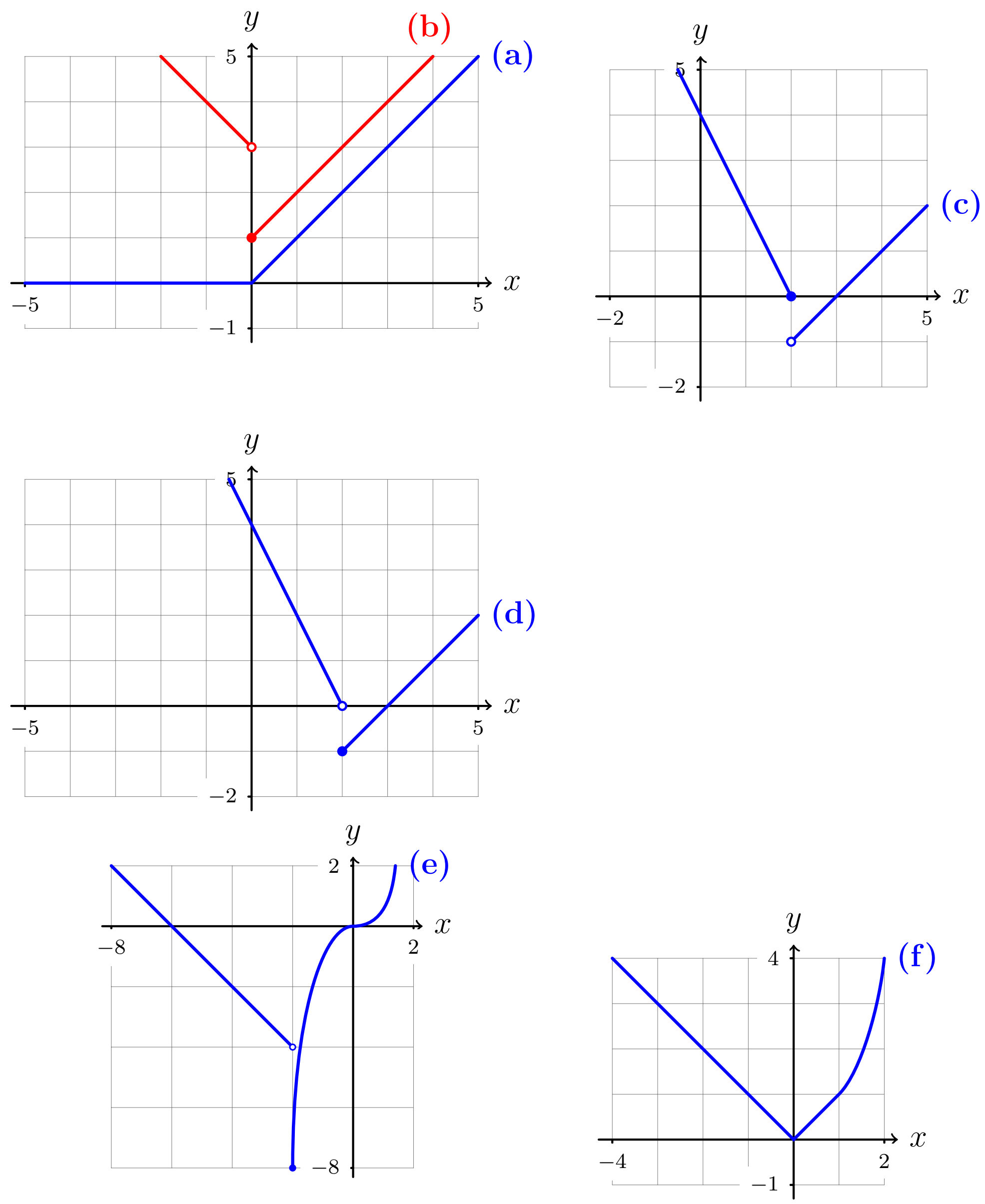
for $\sqrt{4-\sqrt{1+t}}$ to exist. The first of these states that $t \geqslant -1$, and the second is equivalent to $4 \geqslant \sqrt{1+t}$, which is equivalent to $16 \geqslant 1+t$, which is equivalent to $t \leqslant 15$. So altogether the requirement is that

$$-1 \leqslant t \leqslant 15.$$

So the domain is $[-1,15]$.

(h) The domain is $[-1,0) \cup (0,15]$.

3.



4. **(a)** $f(x) = \begin{cases} 1-x, & 0 \leqslant x \leqslant 1 \\ x-1, & 1 < x \leqslant 3 \end{cases}$ and the range is $[0, 2]$.

(b) $f(x) = \begin{cases} x+2, & -2 \leqslant x < -1 \\ -x, & -1 \leqslant x < 1 \\ x-2, & 1 \leqslant x \leqslant 2 \end{cases}$ and the range is $[-1, 1]$.

(c) *Worked Solution:* In the range $-2 \leqslant x \leqslant 0$ the graph is the line with gradient $3/2$ and y-intercept 3, so the function is given by $\frac{3}{2}x + 3$ in this range. And in the range $0 \leqslant x \leqslant 2$ the graph is the line with gradient $-3/2$ and y-intercept 3, so the function is given by $3 - \frac{3}{2}x$ in this range. So we have the following:

$$f(x) = \begin{cases} \frac{3}{2}x + 3, & -2 \leqslant x < 0 \\ 3 - \frac{3}{2}x, & 0 \leqslant x \leqslant 2 \end{cases}$$

We have chosen $x < 0$ in the first line to avoid defining $f(x)$ twice. But we could just as well have written the following, and the answer would still be right.

$$f(x) = \begin{cases} \frac{3}{2}x + 3, & -2 \leqslant x < 0 \\ 3 - \frac{3}{2}x, & 0 \leqslant x \leqslant 2 \end{cases}$$

By looking at which y-values are attained by the function we see that the range is $[0, 3]$.

(d) $f(x) = \begin{cases} -2, & -2 \leqslant x < 0 \\ 2, & 0 \leqslant x < 2 \end{cases}$ and the range is $[-2, 2)$.

5. **(a)** The function is odd. It is not even.

(b) The function is neither odd nor even.

(c) The function is even. It is not odd.

(d) The function is even. It is not odd.

(e) The function is both odd and even.

6. **(a)** The function is even. It is not odd.

(b) The function is odd. It is not even.

(c) The function is neither even nor odd.

(d) *Worked Solution:* $f(x) = \dfrac{x^2}{x^3 + x}$ so that

$$f(-x) = \frac{(-x)^2}{(-x)^3 - x} = \frac{x^2}{-x^3 - x} = -\frac{x^2}{x^3 + x} = -f(x)$$

and consequently f is odd. Since $f(-x) \neq f(x)$, is it not even.

(e) The function is even. It is not odd.

(f) The function is neither even nor odd.

(g) The function is odd. It is not even.

(h) *Worked Solution:* $g(x) = |x| \cdot x^2$ so that

$$g(-x) = |-x| \cdot (-x)^2.$$

Now $|-x| = |x|$ and $(-x)^2 = (-x)(-x) = x^2$, so that

$$g(-x) = |-x| \cdot (-x)^2 = |x| \cdot x^2 = g(x).$$

Thus g is even. Since $g(x) \neq -g(x)$, it is not odd.

(i) The function is odd. It is not even.

(j) *Worked Solution:* We work out $(p+q)(-x)$ and see whether it is equal to $(p+q)(x)$ or $-(p+q)(x)$. We find that

$$(p + q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p + q)(x)$$

so that $p+q$ is even. It is not (necessarily) odd.

(k) The function is odd. It is not (necessarily) even.

(l) The function is odd.

(m) *Worked Solution:* Since q is even and r is odd we have

$$(q \circ r)(-x) = q(r(-x)) = q(-r(x)) = q(r(x))$$

so that $q \circ r$ is even.

(n) The function is even.

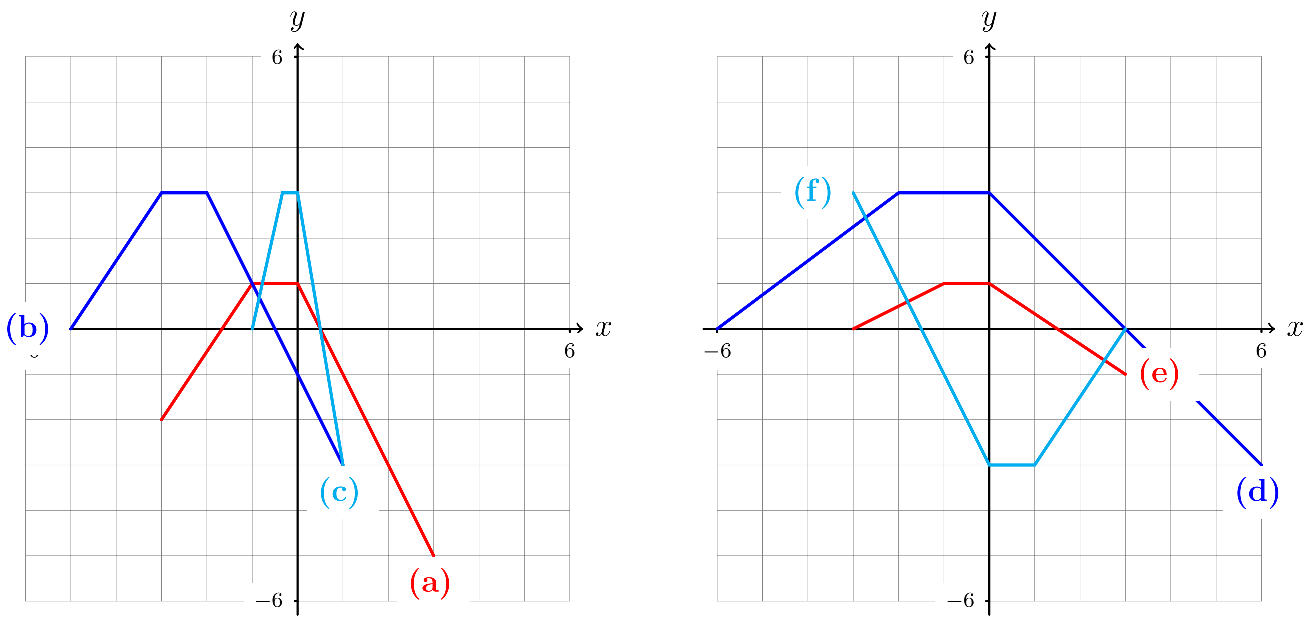
(o) The function is odd.

7. *Worked Solution for g:* Recall that $g(x) = \frac{f(x)+f(-x)}{2}$, so that

$$g(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(-x)+f(x)}{2} = \frac{f(x)+f(-x)}{2} = g(x),$$

and consequently g is even.

8.



9. **(a)** $g(x) = f(x-2) - 1$.

(b) *Worked Solution:* The graph of $h(x)$ is obtained from the graph of f by the following steps: First, reflect the graph of f in the x-axis. Second, move it 2 to the right. Third, move it 1 down. After the first step, we have the graph of the function $-f(x)$. After the second step, we have the graph of the function $-f(x-2)$. After the third step, we have the graph of the function $-f(x-2)-1$. So $h(x) = -f(x-2)-1$.

(c) $k(x) = -\frac{1}{3}f(-x-2)$.

10. **(a)** $(f+g)(x) = x^3 + x^2 - 2$, and its domain is $\mathbb{R}$.

(b) $(f-g)(x) = x^3 - x^2 + 6$, and its domain is $\mathbb{R}$.

(c) $(fg)(x) = (x^3+2)(x^2-4)$, and its domain is $\mathbb{R}$.

(d) *Worked Solution:*

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x^3+2}{x^2-4}$$

Its domain is the collection of all points x that are in the domain of f, and the domain of g, and that satisfy $g(x) \neq 0$. The domain of f and g are both $\mathbb{R}$. And $g(x) = 0$ if and only if $x = 2$ or $x = -2$. So the domain of f/g is $\{x \mid x \neq 2, -2\}$ or in other words $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

11. **(a)** *Worked Solution:* $(f+g)(x) = f(x) + g(x) = \sqrt{5-x} + \sqrt{x^2-4}$. The domain of $f+g$ is the set of all x that lie in the domain of f and the domain of g. The domain of f is the set of points x such that $5 - x \geqslant 0$, or in other words $(-\infty, 5]$. And the domain of g is the set of points x such that $x^2 - 4 \geqslant 0$, or in other words $(-\infty, -2] \cup [2, \infty)$. The intersection of the two domains is then the intersection of $(-\infty, 5]$ with $(-\infty, -2] \cup [2, \infty)$, and this is exactly $(-\infty, -2] \cup [2, 5]$.

(b) $(f-g)(x) = \sqrt{5-x} - \sqrt{x^2-4}$. The domain is $(-\infty, -2] \cup [2, 5]$.

(c) $(fg)(x) = \sqrt{5-x}\sqrt{x^2-4}$. The domain is $(-\infty, -2] \cup [2, 5]$.

(d) $(f/g)(x) = \frac{\sqrt{5-x}}{\sqrt{x^2-4}}$. The domain is $(-\infty, -2) \cup (2, 5]$.

12. In every part, the domain is $\mathbb{R}$.

(a) $(f \circ g)(x) = 3x^2 - 1$

(b) *Worked Solution:* $(g \circ f)(x) = g(f(x)) = g(3x+2) = (3x+2)^2 - 1 = 9x^2 + 12x + 3$. The domain consists of all x such that $f(x)$ is defined and $g(f(x))$ is defined. Since both g and f are defined for all x, it follows that $g \circ f$ is too, hence its domain is $\mathbb{R}$.

(c) $(g \circ g)(x) = x^4 - 2x^2$.

(d) $(f \circ f)(x) = 9x + 8$.

13. In every part, the domain is $\mathbb{R}$.

(a) $(f \circ g)(x) = \sin(2x+2)$.

(b) $(g \circ f)(x) = 2\sin x + 2$.

(c) $(g \circ g)(x) = g(g(x)) = g(2x+2) = 2(2x+2) + 2 = 4x + 6$. Its domain consists of all x for which $g(x)$ and $g(g(x))$ are defined. Since $g(x)$ is defined for all x, it follows that $g \circ g$ is as well. Hence its domain is $\mathbb{R}$.

(d) $(f \circ f)(x) = \sin(\sin(x))$.

14. **(a)** $(f \circ g)(x) = x + 2$, and its domain is $\{x \mid x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$.

(b) *Worked Solution:* $(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)+2} = \frac{1}{\frac{1}{x}+2} = \frac{x}{2x+1}$. Its domain consists of all x for which $f(x)$ and $g(f(x))$ are defined. In other words, it consists of all x for which $x \neq 0$ and $1/x \neq -2$. And that means that it consists of all x for which $x \neq 0$ and $x \neq -1/2$. So the domain is $\{x \mid x \neq 0, x \neq -1/2\} = (-\infty, -1/2) \cup (-1/2, 0) \cup (0, \infty)$.

(c) $(g \circ g)(x) = \frac{x+2}{2x+5}$. Its domain is $\{x \mid x \neq -2, x \neq -5/2\}$.

(d) $(f \circ f)(x) = x$, and its domain is $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

15. **(a)** $f(8)$ does not exist because f is not defined at 8.

(b) $\lim_{x \to 4^-} f(x)$ does not exist because we may find $x < a$ arbitrarily close to a such that $f(x) = 6$, *and* we may find $x < a$ arbitrarily close to a such that $f(x) = 2$.

(c) $\lim_{x \to 4^+} f(x) = 4$.

(d) $\lim_{x \to 4} f(x)$ does not exist for the same reasons that $\lim_{x \to 4^-} f(x)$ does not exist.

(e) $\lim_{x \to 0^+} f(x) = 2$.

(f) $\lim_{x \to 0^-} f(x) = 6$.

(g) $\lim_{x \to 0} f(x)$ does not exist because $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

(h) $f(0) = 2$.

(i) $\lim_{x \to -4^-} f(x) = 2$.

(j) $\lim_{x \to -4^+} f(x) = 4$.

(k) $f(-4) = 3$.

(l) $\lim_{x \to 8} f(x) = 6$.

16. **(a)** $\lim_{x \to -4^-} f(x) = \infty$.

(b) $\lim_{x \to -4^+} f(x) = -\infty$.

(c) $\lim_{x \to -4} f(x)$ does not exist.

(d) $\lim_{x \to 4^-} f(x) = -\infty$.

(e) $\lim_{x \to 4^+} f(x) = -\infty$.

(f) $\lim_{x \to 4} f(x) = -\infty$.

17. **(a)** *Worked Solution:* We use the fact that $\lim_{x \to -1} h(x)$ exists if and only if $\lim_{x \to -1^-} h(x)$ and $\lim_{x \to -1^+} h(x)$ both exist and are equal, in which case

$$\lim_{x \to -1} h(x) = \lim_{x \to -1^-} h(x) = \lim_{x \to -1^+} h(x).$$

Since $h(x) = 2 + x$ for $x < -1$ we have

$$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} 2 + x = 2 + (-1) = 1$$

and since $h(x) = x^2$ for $-1 < x < 1$ we have

$$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} x^2 = (-1)^2 = 1.$$

Since these limits both exist and are equal, we have

$$\lim_{x\to -1} h(x) = 1.$$

(b) The limit does not exist.

(c) *Worked Solution:*

$\lim_{x\to a} f(x)$ exists for all a except possibly $x = -1$ and $x = 1$.

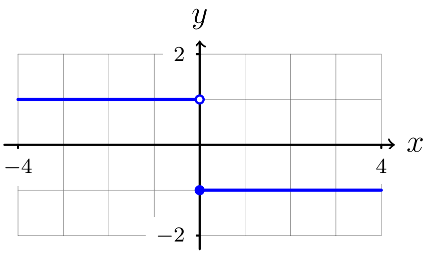
Now $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(-2 - x) = -1$ and $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(x^3) = -1$. Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$, it follows that $\lim_{x\to -1} f(x)$ exists.

Next, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 = 1$, while $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1 - x = 0$. Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, it follows that $\lim_{x\to 1} f(x)$ does not exist.

So $\lim_{x\to a} f(x)$ exists for all a except $a = 1$.

(d) The limit $\lim_{x\to a} f(x)$ exists for all x except $a = 0$.

18. **(a)**



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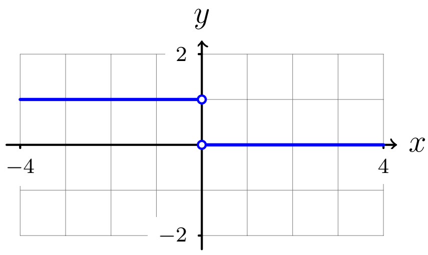
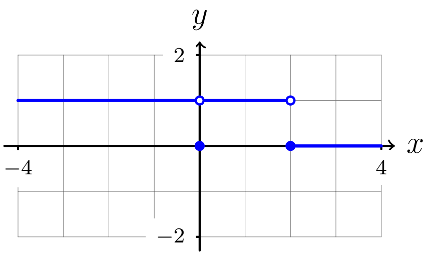
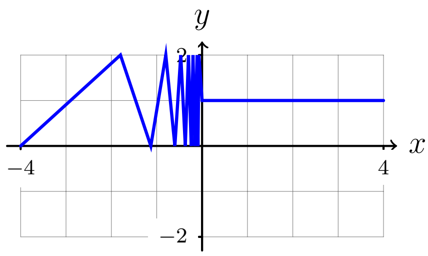


(c)



(d)



19. **(a)** $\displaystyle\lim_{x\to\pi/2^-} \tan x = \infty$ **(b)** $\displaystyle\lim_{x\to\pi/2^+} \tan x = -\infty$ **(c)** $\displaystyle\lim_{x\to 3^+} \frac{1+x}{x-3} = \infty$

(d) $\displaystyle\lim_{x\to 3^-} \frac{1+x}{x-3} = -\infty$ **(e)** $\displaystyle\lim_{x\to 0} \frac{1+x}{x^2(x-3)} = -\infty$

(f) *Worked Solution:*

As x approaches 3 the terms $1+x$ and x^2 approach 4 and 9 respectively, while $x-3$ approaches 0. Thus $\frac{1+x}{x^2(x-3)}$ grows without bound as x approaches 3.

However, if $x > 3$ then the term $1+x$ is positive, as are x^2 and $x-3$. Thus $\frac{1+x}{x^2(x-3)}$ is always positive for $x > 3$.

It follows that $\displaystyle\lim_{x\to 3^+} \frac{1+x}{x^2(x-3)} = \infty.$

20. **(a)** 6

(b) 16

(c) $-\sqrt[3]{3}$

(d) *Worked Solution:* First, $\lim_{x\to 1} 4h(x) = 4\lim_{x\to 1} h(x) = 4 \times -3 = -12$, by the law for functions multiplied by a scalar. Since this limit exists and is nonzero, and since $\lim_{x\to 1} f(x)$ also exists, we may use the limit law for quotients to see that $\lim_{x\to 1} \frac{f(x)}{4h(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} 4h(x)} = \frac{2}{-12} = -1/6$.

(e) 0

21. **(a)** 31

(b) 15

(c) *Worked Solution:* For $u = 3$ we have $u^3 - 3u + 3 = 21$. So for u close to 3 we have $u^3 - 3u + 3 > 0$, and so we may use the n-th root law to see that

$$\lim_{u\to 3} \sqrt{u^3 - 3u + 3} = \sqrt{\lim_{u\to 3}(u^3 - 3u + 3)}$$

and by the direct substitution law this is equal to $\sqrt{3^3 - 3u + 3} = \sqrt{21}$.

(d) $\sqrt{\frac{3}{5}}$

22. The left hand side of the first equation is only defined for $x \neq 3$, while the right hand side is defined for all x. This is nevertheless enough to show that the two limits are equal.

23. **(a)** 1

(b) Does not exist.

(c) 7/4

(d) Does not exist.

(e) -4

(f) 12

(g) Does not exist.

(h) $1/\sqrt{2}$

(i) *Worked Solution:* First we simplify the function for $t \neq 0$:

$$\frac{1}{t} - \frac{2}{t^2 + 2t} = \frac{t+2}{t^2 + 2t} - \frac{2}{t^2 + 2t} = \frac{t + 2 - 2}{t^2 + 2t} = \frac{1}{t+2}$$

Thus

$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{2}{t^2 + 2t}\right) = \lim_{t\to 0} \left(\frac{1}{t+2}\right) = \frac{1}{2}.$$

(j) 1/54

24. *Worked Solution:*

(a) We will show that the left and right handed limits both exist and are zero. Indeed, since $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x \leqslant 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

and

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

so that $\lim_{x\to 0} |x|$ exists and is equal to 0.

(b) We do this in three parts, depending on the value of x.

First suppose that $x = 0$. Then $-|x| = 0$, $f(x) = 0$ and $|x| = 0$. So then $-|x| \leqslant f(x) \leqslant |x|$ certainly holds.

Next suppose that $x > 0$. In this case $|x| = x$. Then since $-1 \leqslant \cos(1/x) \leqslant 1$ and $x > 0$, we have $-x \leqslant x\cos(1/x) \leqslant x$. In other words, $-|x| \leqslant f(x) \leqslant |x|$.

Finally suppose that $x < 0$. In this case $|x| = -x$. Since $-1 \leqslant \cos(1/x) \leqslant 1$ and since $x < 0$, multiplying the inequality by x reverses the inequalities, so we have $-x \geqslant x\cos(1/x) \geqslant x$, or in other words $x \leqslant x\cos(1/x) \leqslant -x$, or in other words $-|x| \leqslant f(x) \leqslant |x|$, as required.

(c) We know that $\lim_{x\to 0}(-|x|) = 0$, $\lim_{x\to 0} |x| = 0$, and $-|x| \leqslant f(x) \leqslant |x|$ for all x. So the squeeze theorem applies and shows us that $\lim_{x\to 0} f(x) = 0$ as required.

25. **(a)** *Worked Solution:* Let us define functions p and q by $p(x) = x^2 - 5x + 8$ and $q(x) = 2x^2 - 11x + 17$. Then the question tells us that $p(x) \leqslant f(x) \leqslant q(x)$. Since p and q are polynomials, direct substitution shows that $\lim_{x\to 3} p(x) = p(3) = 2$ and $\lim_{x\to 3} q(x) = q(3) = 2$. So the squeeze theorem tells us that $\lim_{x\to 3} f(x) = 2$.

(b) 1

26. **(a)** Does not exist.

(b) 3

(c) *Worked Solution:* For $x < 0$ we have $\frac{|x|-3}{x+3} = \frac{-x-3}{x+3} = -1$ so that $\lim_{x\to -3}\left(\frac{|x|-3}{x+3}\right) = \lim_{x\to -3} -1 = -1$.

28. *Worked Solution:* We have $\lim_{x\to 0}\left(\dfrac{g(x)}{x^2}\right) = \lim_{x\to 0}\left(x \cdot \dfrac{g(x)}{x^3}\right) = \lim_{x\to 0}(x) \cdot \lim_{x\to 0}\left(\dfrac{g(x)}{x^3}\right) = 0 \cdot 4 = 0$. Similarly, $\lim_{x\to 0} \frac{g(x)}{x} = 0$.

29. $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$ serve as answers to both parts.

30. The example asks you to show that $\lim_{x\to a} f(x) = L$ where $a = 2$, $f(x) = 2x + 3$ and $L = 7$. So given $\epsilon > 0$ we must specify a $\delta > 0$ and show that if $|x - 2| < \delta$ then $|(2x + 3) - 7| < \epsilon$.

So the error is that the solution proves $|(2x + 3) - 7| < 2\epsilon$. To make a correct solution we can choose a different value of δ and follow through the same reasoning. In this case $\delta = \epsilon/2$ works. (If you can't see why this works, then try defining $\delta = c \cdot \epsilon$ and working through the steps of the solution, then check that by choosing $c = 1/2$ the computation ends as "$< \epsilon$".

31. The question asks us to show that $\lim_{x\to a} f(x) = L$, where $a = 3$, $f(x) = 3 - 5x$, and $L = -12$. So given $\epsilon > 0$, we must define a $\delta > 0$ and show that when $|x - a| < \delta$ we have $|f(x) - L| < \epsilon$.

The solution seems to do this. However there are two wrong steps: first, $|(-5)(3-x)| = |-5| \times |3 - x| = 5|3 - x|$, so there is a sign error in the third $=$, and $|3 - x| = |x - 3|$, so there is another sign error in the fourth $=$. Nothing else needs to be changed.

32. **(c)** *Pre-Solution:* Given $\epsilon > 0$ we must find $\delta > 0$ such that:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \left|(\frac{1}{2}x - 3) - (-2)\right| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \left|\frac{1}{2}x - 1\right| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \frac{1}{2}\,|x - 2| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } |x - 2| < 2\epsilon$$

If we choose $\delta = 2\epsilon$ then this will certainly be true.

Solution: Given $\epsilon > 0$, let $\delta = 2\epsilon$. Then if $0 < |x - 2| < \delta$, we have

$$\left|\left(\frac{1}{2}x - 3\right) - (-2)\right| = \left|\frac{1}{2}x - 1\right| = \frac{1}{2}\,|x - 2| < \frac{1}{2}\delta = \frac{1}{2}2\epsilon = \epsilon.$$

Thus $\lim\limits_{x\to 2} \left(\frac{1}{2}x - 3\right) = -2$ as claimed.

33. **(a)** *Solution:*

Given $\epsilon > 0$, let $\delta = \epsilon$. Then if $0 < |x - a| < \delta$, we have $|x - a| < \delta = \epsilon$. So $\lim\limits_{x\to a} x = a$.

(f) *Pre-Solution:* Given $\epsilon > 0$, we must find $\delta > 0$ such that:

$$\text{if } 0 < |x - (-5)| < \delta, \text{ then } |x^2 - 25| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x + 5| < \delta, \text{ then } |(x - 5)(x + 5)| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x+5| < \delta, \text{ then } |x-5| \cdot |x+5| < \epsilon$$

So we have to choose δ so that if $|x+5| < \delta$ then $|x-5|$ *and* $|x+5|$ are small enough. Let's suppose that $\delta \leqslant 1$. Then if $0 < |x+5| < \delta$, we have $0 < |x+5| < 1$, so that $-1 < x+5 < 1$, and consequently $-11 < x-5 < -9$, and consequently $|x-5| < 11$.

So we know that if $\delta \leqslant 1$ and $0 < |x-(-5)| < \delta$, then $|x^2-25| = |x-5| \cdot |x+5| < 11\delta$. Then if $\delta \leqslant \epsilon/11$, that will be enough. We can arrange this by taking $\delta = \min(1, \epsilon/11)$.

Solution: Given $\epsilon > 0$, let $\delta = \min(1, \epsilon/11)$. Suppose that $0 < |x-(-5)| < \delta$, or in other words that $0 < |x+5| < \delta$. Then $|x+5| < 1$, so that $-1 < x+5 < 1$, and consequently $-11 < x-5 < -9$, and consequently $|x-5| < 11$. So

$$|x^2-25| = |(x-5)(x+5)| = |x-5| \cdot |x+5| < 11 \cdot \delta \leqslant 11 \cdot \epsilon/11 = \epsilon.$$

Or in other words

$$|x^2-25| < \epsilon.$$

We have shown that $\lim\limits_{x \to -5} x^2 = 25$ as required.

34. **(a)** If $a \leqslant b$ then $\min(a,b) = a$. Since $a \leqslant a$ and $a \leqslant b$, we have $\min(a,b) \leqslant a$ and $\min(a,b) \leqslant b$ as required. If $b < a$ then a similar argument shows that the same inequalities hold.

The inequalities are used to show that $\delta \leqslant 2$ in the second paragraph, and to show that $\delta \leqslant 3\epsilon$ in the long series of inequalities.

(b) The identity is used to show that $|2-x| = |-(x-2)| = |x-2|$.

(c) If $|x-2| < 1$ then $-1 < x-2 < 1$, and adding 3 to all terms gives $2 < x+1 < 4$, so that $|x+1| > 2$ and consequently $\frac{1}{|x+1|} < \frac{1}{2}$.

If $|x-2| < 3$ then we find that $|x+1| > 0$, but then we can conclude nothing about $\frac{1}{|x+1|}$.

(d) Given $\epsilon > 0$, define $\delta = \min(1, 6\epsilon)$. Suppose that $0 < |x-2| < \delta$.

Since $|x-2| < \delta$ and $\delta \leqslant 1$, we have $|x-2| < 1$. It follows that $-1 < x-2 < 1$. By adding 3 to all terms we find that $2 < x+1 < 4$. Consequently $|x+1| > 2$, and rearranging gives $\frac{1}{|x+1|} < \frac{1}{2}$.

Now

$$\begin{aligned}\left|\frac{1}{x+1}-\frac{1}{3}\right| &= \left|\frac{3-(x+1)}{3(x+1)}\right| \\ &= \left|\frac{2-x}{3(x+1)}\right| \\ &= \frac{1}{3}\cdot\frac{1}{|x+1|}\cdot|2-x| \\ &= \frac{1}{3}\cdot\frac{1}{|x+1|}\cdot|x-2| \\ &< \frac{1}{3}\cdot\frac{1}{2}\cdot\delta \\ &\leqslant \frac{1}{3}\cdot\frac{1}{2}\cdot 6\epsilon \\ &= \epsilon\end{aligned}$$

and so $\left|\dfrac{1}{x+1}-\dfrac{1}{3}\right| < \epsilon$. Thus $\lim_{x\to 2}(\frac{1}{x+1}) = \frac{1}{3}$ as required.

(e) In this case you cannot prove an inequality of the form $\frac{1}{|x+1|} <?$ and so the solution cannot be made to work in this case.

35. **(a)** *Solution:*

Let $\epsilon > 0$. Define $\delta = \min(1, \epsilon/4)$ and let x be such that $0 < |x-2| < \delta$.

Since $\delta = \min(1, \epsilon/4)$ it follows that $\delta \leqslant 1$ so that $|x-2| < 1$. Thus $-1 < x-2 < 1$, so that $2 < x+1 < 4$, so that $|x+1| < 4$.

Now

$$\begin{aligned}|(x^2-x-3)-(-1)| &= |x^2-x-2| \\ &= |(x+1)(x-2)| \\ &= |x+1|\cdot|x-2| \\ &< 4\cdot\delta \\ &\leqslant 4\cdot\epsilon/4 \\ &= \epsilon\end{aligned}$$

so that overall we have $|(x^2-x-3)-(-1)| < \epsilon$ as required.

(d) *Solution:*

Let $\epsilon > 0$. Define $\delta = \min(1, \frac{12\epsilon}{5})$ and let x be such that $0 < |x-1| < \delta$.

Since $\delta = \min(1, \frac{12\epsilon}{5})$ it follows that $\delta \leqslant 1$, so that $|x-1| < 1$, so that $-1 < x-1 < 1$, so that $3 < x+3 < 5$, so that $|x+3| > 3$ and consequently $\frac{1}{|x+3|} < \frac{1}{3}$.

Now

$$\begin{aligned}\left|\frac{x-2}{x+3}-\left(-\frac{1}{4}\right)\right| &= \left|\frac{x-2}{x+3}+\frac{1}{4}\right| \\ &= \left|\frac{4(x-2)+(x+3)}{4(x+3)}\right| \\ &= \left|\frac{5x-5}{4(x+3)}\right| \\ &= \frac{5}{4}\frac{|x-1|}{|x+3|} \\ &< \frac{5}{4}|x-1|\frac{1}{3} \\ &< \frac{5}{4}\cdot\frac{12\epsilon}{5}\cdot\frac{1}{3} \\ &= \epsilon\end{aligned}$$

so that overall $\left|\frac{x-2}{x+3}-\left(-\frac{1}{4}\right)\right| < \epsilon$ as required.

36. *Worked Solution:*

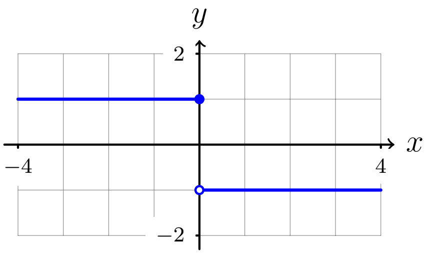
(a) -4 is a discontinuity of f because $f(-4)$ is not defined.

(b) 4 is a discontinuity of f because $\lim\limits_{x\to 4} f(x)$ does not exist, which is because $\lim\limits_{x\to 4^-} f(x)$ does not exist.

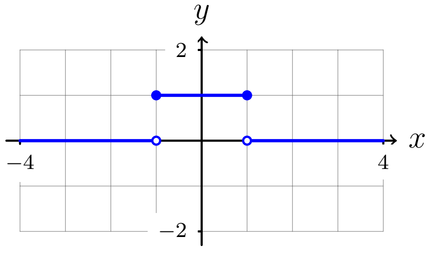
(c) 8 is a discontinuity of f because $\lim_{x\to 8} f(x) = 6$ but $f(8) = 2$.

(d) f is defined at all numbers x in the range $-10 \leqslant x \leqslant 10$, except for $x = -4$. It is continuous at all of those points except for $x = 4$ and $x = 8$. So it is continuous on the intervals $[-10, -4)$, $(-4, 4)$, $(4, 8)$ and $(8, 10]$.

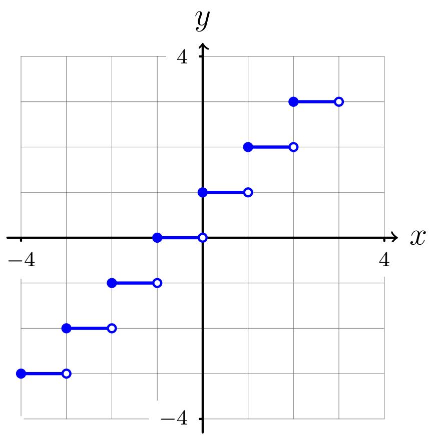
37. **(a)**



(b)



(c)



38. **(a)** Because $\lim_{x\to 2} f(x) = 0$ but $f(2) = 1$.

(b) Because $\lim_{x\to 0} f(x)$ does not exist.

(c) *Worked Solution:* For f to be continuous at $a = 0$ we require that:

1. $f(0)$ is defined.
2. $\lim_{x\to 0} f(x)$ exists.
3. $\lim_{x\to 0} f(x) = f(0)$.

The first condition holds because $f(0)$ is defined to be 1. And the second condition holds because

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin x = \sin 0 = 0$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} x^2 = 0^2 = 0$$

are equal. However, the third condition does not hold because $\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$. Since the third condition doesn't hold, it follows that f is not continuous at 0.

39. **(a)** Discontinuous at 0, and continuous at neither right nor left there.

(b) No discontinuities.

(c) Discontinuous at 0, and continuous from the right but not the left there. Discontinuous at 1, and continuous from the left but not from the right there.

(d) Discontinuous at 0. Continuous at 0 from the right, but not the left. (It is continuous at 1.)

40. **(a)** *Worked Solution:* For f to be continuous at 1 we require that $f(1)$ is defined, that $\lim_{x\to 1} f(x)$ exists, and that $\lim_{x\to 1} f(x) = f(1)$.

- Certainly $f(1) = c \cdot 1^2 - 1 = c - 1$ is defined.

- Now we check when $\lim_{x\to 1} f(x)$ exists. We have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+1 = 2$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} cx^2 - x = c-1$. So for the limit to exist we need $2 = c-1$, or in other words that $c = 3$.
- Finally, when the limit does exist, i.e. when $c = 3$, we have that $\lim_{x\to 1} f(x) = 2 = f(1)$.

So the final answer is that f is continuous at 1 when $c = 3$.

(b) The function is clearly continuous at every real number except perhaps -1. In order for f to be continuous at -1, we need that $f(-1)$ is defined, that $\lim\limits_{x\to -1} f(x)$ exists, and that $\lim\limits_{x\to -1} f(x) = f(-1)$. Certainly $f(-1)$ is defined; the limit exists if and only if

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$$

or in other words

$$2(-1)^2 + b(-1) = -2 + b$$

or in other words

$$b = 2;$$

and in this case the third condition holds.

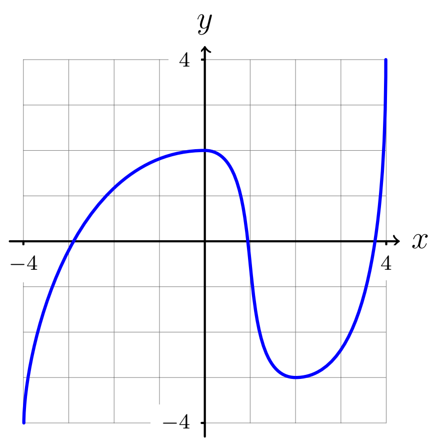
41. **(b)** *Worked Solution:*

Define f by $f(x) = \sin(x) - 2\cos(x)$. It is continuous on $(-\infty, \infty)$, so it is continuous on $[-\pi/2, \pi/2]$. Also

$$f(-\pi/2) = \sin(-\pi/2) - 2\cos(-\pi/2) = -1 - 2\cdot 0 = -1,$$
$$f(\pi/2) = \sin(\pi/2) - 2\cos(\pi/2) = 1 - 2\cdot 0 = 1.$$

So $f(-\pi/2) < N < f(\pi/2)$ where $N = 0$. The intermediate value theorem tells us that there is $c \in (-\pi/2, \pi/2)$ such that $f(c) = N$, or in other words such that $\sin(c) - 2\cos(c) = 0$.

42.



43. **(a)** $f'(a) = 0$. **(b)** $f'(a) = 1$. **(c)** $f'(a) = 2a$.

(d) *Worked Solution:*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{1/(a+h) - 1/a}{h} \\ &= \lim_{h\to 0} \frac{(a-(a+h))/a(a+h)}{h} \\ &= \lim_{h\to 0} \frac{-h/a(a+h)}{h} \\ &= \lim_{h\to 0} -1/a(a+h) \\ &= -1/a^2 \end{aligned}$$

44. **(a)** *Worked Solution:* We calculate

$$F'(2) = \lim_{h\to 0} \frac{\frac{3}{(2+h)^2+2+h} - \frac{3}{2^2+2}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{3}{h^2+5h+6} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{6 - h^2 - 5h - 6}{2h(h^2+5h+6)} =$$

$$\lim_{h\to 0} \frac{-h-5}{2(h^2+5h+6)} = \frac{-5}{12}.$$

(b) *Worked Solution:* The tangent line to the curve $y = F(x)$ at the point $(2, 1/2)$ has equation $y - 1/2 = F'(2)(x-2)$, which simplifies to give $y = -\frac{5x}{12} + \frac{4}{3}$.

45. **(a)** $f'(a) = 4a - 3$

(b) *Worked Solution:*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{[(a+h)^3 - 2] - [a^3 - 2]}{h} \\ &= \lim_{h\to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2 - a^3 + 2}{h} \\ &= \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\ &= \lim_{h\to 0} 3a^2 + 3ah + h^2 \\ &= 3a^2 \end{aligned}$$

(c) $f'(a) = -\dfrac{1}{(2a+3)^2}$

(d) $f'(a) = \dfrac{-1}{2\sqrt{2-a}}$

(e) *Worked Solution:*

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{a+h+1}} - \frac{1}{\sqrt{a+1}}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{a+1}-\sqrt{a+h+1}}{\sqrt{a+h+1}\sqrt{a+1}}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(\sqrt{a+1}-\sqrt{a+h+1})(\sqrt{a+1}+\sqrt{a+h+1})}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+h+1})}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(a+1)-(a+h+1)}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+h+1})}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{-h}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+h+1})}\right] \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+h+1})} \\
&= \frac{-1}{\sqrt{a+0+1}\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+0+1})} \\
&= \frac{-1}{2\sqrt{a+1}^3}
\end{aligned}$$

46. *Worked Solution:*

$$g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = \lim_{h\to 0} \frac{f(a+h)+c-f(a)-c}{h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

47. This question can be answered quickly using the solutions to an earlier question about the same functions.

(a) $f'(x) = 0$. The domain of f and of f' are both $\mathbb{R}$.

(b) $f'(x) = 1$. The domain of f and of f' are both $\mathbb{R}$.

(c) $f'(x) = 2x$. The domain of f and of f' are both $\mathbb{R}$.

(d) $f'(x) = \frac{-1}{x^2}$. The domain of f and of f' are both $\{x \mid x \neq 0\}$.

48. **(a)** $f'(x) = 1 - \frac{1}{2\sqrt{x}}$. The domain of f is $[0,\infty)$ and the domain of f' is $(0,\infty)$.

(b) *Worked Solution:* The domain of g is $(-\infty, 0]$. The derivative of g is:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\sqrt{-x-h}-\sqrt{-x}}{h} \\ &= \lim_{h\to 0} \frac{(-x-h)-(-x)}{h(\sqrt{-x-h}+\sqrt{-x})} \\ &= \lim_{h\to 0} \frac{-1}{\sqrt{-x-h}+\sqrt{-x}} \\ &= \frac{-1}{2\sqrt{-x}} \end{aligned}$$

It exists for $x \leqslant 0$ and $x \neq 0$, so the domain of g' is $(-\infty, 0)$.

(c) The derivative f' is given by $f'(x) = \dfrac{-12}{(2+3x)^2}$. The domain of f and f' are both equal to $\{x \mid x \neq \frac{-2}{3}\}$.

49. **(a)** *Worked Solution:*

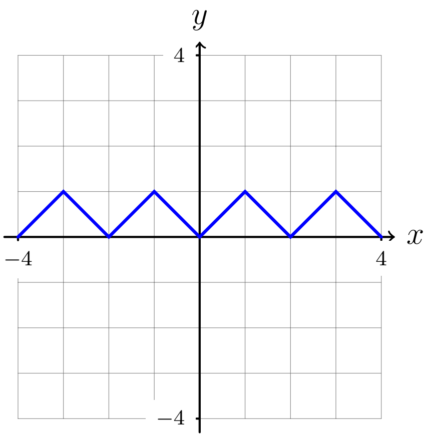
$$f'(0) = \lim_{h\to 0} \frac{h^{\frac{3}{5}}}{h} = \lim_{h\to 0} \frac{1}{h^{\frac{2}{5}}}$$

which does not exist because as h approaches 0, $h^{\frac{2}{5}}$ approaches 0 but 1 does not. So $f'(0)$ does not exist.

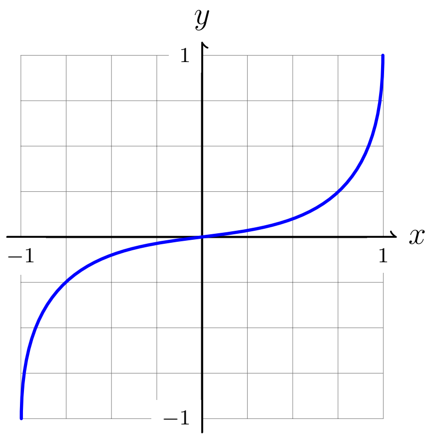
(b) $f'(0)$ exists.

(c) $f'(0)$ exists.

50. **(a)**



(b)



51. It follows that f' is an odd function, because

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{-h} \\ &= -\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= -f'(x). \end{aligned}$$

Here the third equation came from substituting h for $-h$ in the limit. (Why is this valid?)

52. **(a)** $x^{1/n}$ **(b)** x^{ab} **(c)** x^{a+b} **(d)** x^{-a} **(e)** $x^{7/12}$ **(f)** $x^{-17/6}$

53. **(a)** 0. **(b)** 0.

(c) *Worked Solution:* $\dfrac{d}{dx}[3x+2] = \dfrac{d}{dx}[3x] + \dfrac{d}{dx}[2] = 3\dfrac{d}{dx}[x] + \dfrac{d}{dx}[2] = 3\cdot 1 + 0 = 3.$

(d) $-77x^{10}$. **(e)** $6x^2 - 2x$.

(f) *Worked Solution:* $\dfrac{d}{dt}[f(t)] = \dfrac{d}{dt}[\dfrac{1}{3}t^5 - 2t^3 + t^2] = \dfrac{d}{dt}[\dfrac{1}{3}t^5] - \dfrac{d}{dt}[2t^3] + \dfrac{d}{dt}[t^2] = \dfrac{1}{3}\dfrac{d}{dt}[t^5] - 2\dfrac{d}{dt}[t^3] + \dfrac{d}{dt}[t^2] = \dfrac{1}{3}5t^4 - 2\cdot 3t^2 + 2t = \dfrac{5}{3}t^4 - 6t^2 + 2t.$

(g) $-\frac{3}{5}x^{-8/5}$. **(h)** $-3cy^{-4}$.

(i) *Worked Solution:* $\frac{d}{ds}[B(s)] = \frac{d}{ds}[\frac{8}{s^3}] = \frac{d}{ds}[8s^{-3}] = 8\frac{d}{ds}[s^{-3}] = 8\cdot(-3)s^{-4} = -24s^{-4}$.

(j) $\frac{4}{3}r^{1/3} - \frac{1}{3}r^{-2/3}$ **(k)** $1 + \frac{1}{\sqrt{p}}$ **(l)** $2\pi r$ **(m)** $\sqrt{3}(1 - \frac{1}{2\sqrt{u}})$

(n) $\frac{1}{3}u^{-2/3} + \frac{9}{2}u^{1/2}$

54. Both methods give $f'(x) = 8x^3 + 6x^2 - 2x - 1$. I found it quicker to multiply first and then differentiate.

55. Both methods give $f'(x) = 2 - \frac{3}{2}x^{-\frac{5}{2}} + x^{-2}$. It is faster to simplify first.

56. **(a)** $6x^2 - 4x^3$ **(b)** $4x + 1$ **(c)** $\frac{8}{3}x^{1/3} + x^{-2/3}$ **(d)** $8t - 4$

(e) *Worked Solution:* By multiplying out $(x - x^{-1})(x - x^{-1})(x - x^{-1})$, or by using the binomial theorem, we find that $H(x) = x^3 - 3x + 3x^{-1} - x^{-3}$, so that $H'(x) = 3x^2 - 3 - 3x^{-2} + 3x^{-4}$.

(f) $1 - x^{-2}$ **(g)** $-10y^{-3} - 9y^{-2}$ **(h)** $-3s^{-2} - 9s^{-4} - 2s^{-3} - 4s^{-5}$

(i) $\dfrac{2t + t^4}{(1 - t^3)^2}$ **(j)** $-2t^{-3} - 1$

(k) *Worked Solution:*

$$f(x) = \frac{x+2}{x^2+3x+2} = \frac{x+2}{(x+2)(x+1)} = \frac{1}{x+1}$$

and so by the quotient rule

$$f'(x) = \frac{(x+1).0 - (1+0).1}{(x+1)^2} = -\frac{1}{(x+1)^2}.$$

(l) $1 - v^{-2/3}$ **(m)** $\frac{1}{2}t^{-1/2} - \frac{1}{6}t^{-7/6}$ **(n)** $\dfrac{1}{2\sqrt{t}}$ **(o)** $\dfrac{x^4-3x^2}{(x^2-1)^2}$

(p) $2ax + b$

57. **(a)** *Worked Solution:* $y = \dfrac{x}{2x-1}$, so

$$\frac{dy}{dx} = \frac{(2x-1)\cdot 1 - x\cdot(2-0)}{(2x-1)^2} = \frac{2x-1-2x}{(2x-1)^2} = \frac{-1}{(2x-1)^2}.$$

So when $x = 1$ we have $\dfrac{dy}{dx} = \dfrac{-1}{(2\cdot 1-1)^2} = -1$. Thus the equation of the tangent line through $(1,1)$ is $y - 1 = -1(x-1)$ or in other words $y = -x + 2$.

(b) $y = 7x - 3$

58. **(a)** $f'(x) = 4x^3 - 6x^2 + 6x - 4$ and $f''(x) = 12x^2 - 12x + 6$.

(b) *Worked Solution:* Since $G(r) = \sqrt[4]{r} - 2\sqrt[3]{r} = r^{1/4} - 2r^{1/3}$, we have

$$G'(r) = \tfrac{1}{4}r^{\frac{1}{4}-1} - 2\cdot\tfrac{1}{3}r^{\frac{1}{3}-1} = \tfrac{1}{4}r^{-\frac{3}{4}} - \tfrac{2}{3}r^{-\frac{2}{3}}$$

and

$$G''(r) = \tfrac{1}{4}\cdot(-\tfrac{3}{4})r^{-\frac{3}{4}-1} - \tfrac{2}{3}\cdot(-\tfrac{2}{3})r^{-\frac{2}{3}-1} = -\tfrac{3}{16}r^{-\frac{7}{4}} + \tfrac{4}{9}r^{-\frac{5}{3}}$$

(c) $f'(x) = \dfrac{6x^2+2x^3}{(2+x)^2}$ and $f''(x) = 2x\dfrac{x^2+6x+12}{(x+2)^3}$.

(d) $f'(x) = \frac{3}{(2-3x)^2}$ and $f''(x) = \frac{18}{(2-3x)^3}$.

(e) *Worked Solution:* Since $f(x) = \dfrac{1}{x^2-x-1}$ we have by the chain rule:

$$\begin{aligned} f'(x) &= -\frac{1}{(x^2-x-1)^2}\cdot(2x-1) \\ &= -\frac{2x-1}{(x^2-x-1)^2}. \end{aligned}$$

Then by the quotient rule we have:

$$\begin{aligned}
f''(x) &= -\frac{(x^2-x-1)^2\cdot 2-(2x-1)\cdot 2\cdot(2x-1)(x^2-x-1)}{(x^2-x-1)^4}\\
&= -\frac{(x^2-x-1)\cdot 2-(2x-1)\cdot 2\cdot(2x-1)}{(x^2-x-1)^3}\\
&= -\frac{(2x^2-2x-2)-2(4x^2-4x+1)}{(x^2-x-1)^3}\\
&= -\frac{2x^2-2x-2-8x^2+8x-2}{(x^2-x-1)^3}\\
&= -\frac{-6x^2+6x-4}{(x^2-x-1)^3}\\
&= 2\cdot\frac{3x^2-3x+2}{(x^2-x-1)^3}
\end{aligned}$$

59. **(a)** $h'(2) = 3$.

(b) *Worked Solution:* Since $h(x) = f(x)g(x)$, the product rule shows us that $h'(x) = f'(x)g(x) + f(x)g'(x)$. So $h'(2) = f'(2)g(2) + f(2)g'(2) = (-1)\cdot 2 + 3\cdot 0 = -2$.

(c) $h'(2) = 2/9$.

(d) $h'(2) = -1/2$.

(e) *Worked Solution:* Since $h(x) = \frac{f(x)}{1-g(x)}$, the quotient rule gives

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left(\frac{f(x)}{1-g(x)}\right)\\
&= \frac{\frac{d}{dx}[f(x)]\cdot(1-g(x)) - f(x)\cdot\frac{d}{dx}[1-g(x)]}{(1-g(x))^2}\\
&= \frac{f'(x)(1-g(x))+f(x)g'(x)}{(1-g(x))^2}.
\end{aligned}$$

Thus

$$h'(2) = \frac{f'(2)(1-g(2))+f(2)g'(2)}{(1-g(2))^2} = \frac{(-1)\cdot(1-2)+3\cdot 0}{(1-2)^2} = 1.$$

(f) *Worked Solution:* $(q\circ r)'(1) = q'(r(1))\cdot r'(1) = q'(2)\cdot r'(1) = 9\cdot 4 = 36$.

(g) *Worked Solution:* We write $p\circ q\circ r$ as $p\circ(q\circ r)$ and apply the chain rule to get

$$(p\circ q\circ r)'(x) = (p\circ(q\circ r))'(x) = p'((q\circ r)(x))\cdot(q\circ r)'(x) = p'(q(r(x)))\cdot(q\circ r)'(x).$$

Now to find $(q\circ r)'(x)$ we may either use what we were told in the previous part, or we apply the chain rule again, as follows.

$$(p\circ q\circ r)'(x) = p'(q(r(x)))\cdot(q\circ r)'(x) = p'(q(r(x)))\cdot q'(r(x))\cdot r'(x)$$

as required. Thus

$$(p\circ q\circ r)(1) = p'(q(r(1)))\cdot q'(r(1))\cdot r'(1) = p'(q(2))\cdot q'(2)\cdot 4 = p'(3)\cdot 9\cdot 4 = 4\cdot 9\cdot 4 = 144.$$

61. We have $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. So if $f'(x) = 3$ we get $x^{\frac{2}{3}} = \frac{1}{9}$. This equation has the unique solution $x = \frac{1}{27}$. We then get $f(x) = \frac{1}{3}$ so the equation of the tangent line is $y - \frac{1}{3} = 3(x - \frac{1}{27})$ or $y = 3x + \frac{2}{9}$.

63. **(a)** The function is differentiable for all $a \neq \pm 1$.

(b) The function is differentiable for $x \neq 1$ In both cases we prove this by calculating the limit in the definition of the derivative from left and from right.

64. **(a)** *Worked Solution:*

$$\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} = \lim_{\theta \to 0} \frac{1}{\frac{\sin(\theta)}{\theta}} = \frac{1}{\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}} = \frac{1}{1} = 1.$$

(b) $\lim_{\theta \to 0} \frac{\theta^2}{\sin(\theta)} = 0$.

(c) *Worked Solution:* We divide top and bottom of the fraction by θ so that the expression now consists of terms whose limits we know.

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta + \sin(\theta)} = \lim_{\theta \to 0} \frac{\frac{\sin(\theta)}{\theta}}{1 + \frac{\sin(\theta)}{\theta}} = \frac{\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}}{1 + \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}} = \frac{1}{1+1} = 1/2$$

(d) $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta + \theta^2} = 1$.

65. **(a)** $f'(x) = 2\cos x - 3x^2$

(b) $f'(x) = \cos x - x \sin x$

(c) $f'(x) = -\sin x - \frac{1}{2}\csc x \cot x$

(d) $\frac{dy}{dx} = -2\csc x \cot x - \sec x \tan x$

(e) $g'(t) = 2t \sin t + t^2 \cos t$

(f) $g'(t) = \sec t[\sec t - 2\tan t]$

(g) *Worked Solution:* Since $y = a \sin t + t^2 \cos t$, we have

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left[a \sin t + t^2 \cos t\right] \\ &= a\frac{d}{dt}[\sin t] + \frac{d}{dt}[t^2 \cos t] \\ &= a \cos t + t^2 \frac{d}{dt}[\cos t] + \frac{d}{dt}[t^2] \cdot \cos t \\ &= a \cos t - t^2 \sin t + 2t \cos t. \end{aligned}$$

(h) $y' = (a \sin u + b \tan u) + u(a \cos u + b \sec^2 u)$

(i) $y' = \dfrac{2 - \cot x + x \csc^2 x}{(2 - \cot x)^2}$

(j) $\frac{dy}{dx} = \sec x \cdot [\tan^2 x + \sec^2 x]$

(k) $f'(\theta) = \dfrac{\sec^2\theta}{(1+\tan\theta)^2}$

(l) *Worked Solution:*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{\cos x}{1+\sin x}\right] \\
&= \frac{(1+\sin x)\cdot \frac{d}{dx}\cos x - \cos x \cdot \frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} \\
&= \frac{-(1+\sin x)\cdot \sin x - \cos x \cdot \cos x)}{(1+\sin x)^2} \\
&= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\
&= \frac{-\sin x - 1}{(1+\sin x)^2} \\
&= -\frac{1}{1+\sin x}
\end{aligned}$$

(m) $y' = -\dfrac{\sec x + \sec^2 x}{\tan^2 x}$

(n) $\frac{dy}{dx} = x\cos(x)\cot(x)[2 + x\cot(x)]\ x\cos x[2\cot x - x - x\csc^2 x]$

66. *Worked Solution:*

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x\sin h}{h} \\
&= \lim_{h\to 0}\left[\cos x\frac{\cos h - 1}{h} - \sin x\frac{\sin h}{h}\right] \\
&= \cos x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

67. *Worked Solution for* $\sec x$*:*

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = -\frac{\frac{d}{dx}\cos x}{\cos^2 x} = -\frac{-\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \cdot \sec x$$

68. **(a)** $y = \frac{2}{3}x + (\frac{2}{\sqrt{3}} - \frac{\pi}{9})$

(b) $y = x$

(c) *Worked Solution:* Since $y = x + \sec x$, we have $\frac{dy}{dx} = \frac{d}{dx}[x + \sec x] = 1 + \sec x \tan x$, so when $x = \pi$ we have $\frac{dy}{dx} = 1 + \sec \pi \tan \pi = 1 + -1 \cdot 0 = 1$. Thus the equation of the tangent line is $y - (\pi - 1) = 1(x - \pi)$, or in other words $y = x - 1$.

69. *Worked Solution:*

Since $45 = 4 \times 11 + 1$, we have $\frac{d^{45}}{dx^{45}} \sin x = \frac{d}{dx} \sin x = \cos x$.

For the second part, we work out that $\frac{d^4}{dx^4}[x \cos x] = \frac{d^3}{dx^3}[-x \sin x + \cos x] = \frac{d^2}{dx^2}[-x \cos x - 2 \sin x] = \frac{d}{dx}[x \sin x - 3 \cos x] = x \cos x + 4 \sin x$. Thus $\frac{d^8}{dx^8}[x \cos x] = \frac{d^4}{dx^4}[x \cos x + 4 \sin x] = x \cos x + 4 \sin x + 4 \sin x = x \cos x + 8 \sin x$. Proceeding in this way, we find that $\frac{d^{16}}{dx^{16}}[x \cos x] = x \cos x + 16 \sin x$.

70. It has horizontal tangent at the points of the form $(\frac{2\pi}{3} + 2k\pi, \frac{1}{\sqrt{3}})$ and $(-\frac{2\pi}{3} + 2k\pi, -\frac{1}{\sqrt{3}})$.

71. **(a)** *Worked Solution:* $y = \cos(3x) = \cos(g(x)) = f(g(x))$ where $g(x) = 3x$ and $f(u) = \cos(u)$.

Consequently $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$, and since $f'(u) = -\sin(u)$ and $g'(x) = 3$, we have

$$\frac{dy}{dx} = -\sin(g(x)) \cdot 3 = -3\sin(3x).$$

(b) $g(x) = 4 + 3x$, $f(u) = \sqrt{u}$, $\frac{dy}{dx} = \frac{3}{2\sqrt{4+3x}}$

(c) $g(x) = 1 - x^3$, $f(u) = u^5$, $\frac{dy}{dx} = -15x^2(1 - x^3)^4$.

(d) *Worked Solution:* $y = \sqrt[3]{\sin(x)} = \sqrt[3]{g(x)} = f(g(x))$ where $g(x) = \sin(x)$ and $f(u) = \sqrt[3]{u}$.

So $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$. Now $f'(u) = \frac{1}{3u^{2/3}}$ and $g'(x) = \cos(x)$, so that

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = \frac{1}{3(g(x))^{2/3}} \cdot \cos(x) = \frac{\cos(x)}{3(\sin(x))^{2/3}}.$$

72. **(a)** $F'(x) = 16x(x^4 - 2x^2 + 2)^3(x^2 - 1)$

(b) $F'(x) = \dfrac{1 - x^2}{(1 + 3x - x^3)^{2/3}}$

(c) $g'(t) = -\dfrac{12t^2}{(t^3 + 1)^5}$.

(d) $\frac{dy}{dx} = 4x^3 \cos(a^4 + x^4)$

(e) $\frac{dy}{dx} = 4 \sin^3 x \cdot \cos x$

(f) $y' = x\csc(kx)[2 - kx\cot(kx)]$.

(g) $f'(x) = (3x-2)^3(x^4-x-1)^4[72x^4 - 40x^3 - 27x - 2]$

(h) $h'(t) = [34t^2 - 32t - 1]\dfrac{(2t^2-1)^3}{2(t-1)^{1/2}}$

(i) $y' = 12x\dfrac{(x^2-1)^2}{(x^2+1)^4}$

(j) $y' = -\sin(x\sin x)(x\cos x + \sin x)$

(k) $F'(z) = -\dfrac{1}{(z+1)^{1/2}(z-1)^{3/2}}$

(l) $y' = \dfrac{1}{(1-r^2)^{3/2}}$

(m) $y' = \dfrac{x\sin\sqrt{1-x^2}}{\sqrt{1-x^2}}$

(n) $y' = 2\cos(2x)\cdot cos^{-2}(\sin 2x)$

(o) $y' = -16\cos 2x\dfrac{(1-\sin 2x)^3}{(1+\sin 2x)^5}$

(p) $y' = \dfrac{2x^{1/2}-1}{4x^{1/2}(x-x^{1/2})^{1/2}}$

73. **(a)** *Worked Solution:* The first derivative is

$$\frac{dy}{dx} = \cos(x^2)\cdot 2x = 2x\cos(x^2)$$

and the second derivative is

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(2x)\cdot\cos(x^2) + 2x\cdot\frac{d}{dx}\cos(x^2)\\ &= 2\cos(x^2) + 2x\cdot 2x\cdot(-\sin(x^2))\\ &= 2\cos(x^2) - 4x^2\sin(x^2).\end{aligned}$$

(b) $\frac{dy}{dx} = 2\sin(x)\cos(x)$ and $\frac{d^2y}{dx^2} = 2(\cos^2(x) - \sin^2(x))$.

(c) $K'(t) = 5\sec^2(5t)$ and $K''(t) = 50\sec^2(5t)\tan(5t)$

74. *Worked Solution:* The product rule tells us that

$$f'(x) = \frac{d}{dx}(x)\cdot g(x^3) + x\cdot\frac{d}{dx}(g(x^3)) = g(x^3) + x\frac{d}{dx}(g(x^3)),$$

and the chain rule tells us that $\frac{d}{dx}(g(x^3)) = g'(x^3)\cdot 3x^2$. So

$$f'(x) = g(x^3) + 3x^3g'(x^3).$$

Similarly,

$$f''(x) = 3x^2g'(x^3) + 9x^2g'(x^3) + 3x^3\cdot 3x^2\cdot g''(x^3) = 12x^2g'(x) + 9x^5g''(x^3).$$

75. **(a)** $y' = -\dfrac{x^4}{y^4}$

(b) *Worked Solution:* Differentiating both sides of the equation $4\sqrt{x} - 4\sqrt{y} = 1$ yields

$$4\frac{d}{dx}\sqrt{x} - 4\frac{d}{dx}\sqrt{y} = \frac{d}{dx}1$$

i.e.

$$4\frac{1}{\sqrt{x}} - 4\frac{1}{\sqrt{y}}\frac{d}{dx}y = 0$$

i.e.

$$\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}y' = 0.$$

Rearranging this gives

$$y' = \frac{\sqrt{y}}{\sqrt{x}}.$$

(c) $y' = -\dfrac{2x+y}{x+2y}$

(d) $y' = -\dfrac{3x^2+4xy-2y^3}{2x^2-6xy^2}$

(e) *Worked Solution:* The given equation $x^3(x+y) = y^3(2x-y)$ can be written as $x^4 + x^3y = 2xy^3 - y^4$. Differentiating both sides with respect to x gives

$$4x^3 + (3x^2y + x^3y') = (2y^3 + 2x \cdot 3y^2y') - 4y^3y'.$$

Rearranging,

$$4x^3 + 3x^2y + x^3y' = 2y^3 + 6xy^2y' - 4y^3y'.$$
$$4x^3 + 3x^2y - 2y^3 = 6xy^2y' - 4y^3y' - x^3y'$$
$$4x^3 + 3x^2y - 2y^3 = y'(6xy^2 - 4y^3 - x^3)$$

so that

$$y' = \frac{4x^3 + 3x^2y - 2y^3}{6xy^2 - 4y^3 - x^3}.$$

(f) $y' = -\dfrac{y^2 + 2x\sin y}{2xy + x^2\cos y}$

(g) $y' = \dfrac{1 - y^2\sin(xy^2)}{2xy\sin(xy^2)}$

(h) $y' = -\dfrac{\tan x}{\tan y}$

(i) $y' = \dfrac{\cos y^2 + 2xy\sin x^2}{\cos x^2 + 2xy\sin y^2}$

(j) $y' = \dfrac{y(\cos(x/y) - y^2)}{x(y^2 + \cos(x/y))}$

(k) *Worked Solution:* Differentiating both sides of the equation $\sqrt{x-y} = 1 - x^2y^2$ with respect to x gives

$$\frac{1}{2\sqrt{x-y}}\frac{d}{dx}(x-y) = -\frac{d}{dx}(x^2y^2)$$

or in other words

$$\frac{1}{2\sqrt{x-y}}(1-y') = -(2x\cdot y^2 + x^2\cdot 2yy').$$

Rearranging gives the following:

$$\frac{1}{2\sqrt{x-y}} - \frac{y'}{2\sqrt{x-y}} = -2xy^2 - 2x^2yy'$$

$$\frac{1}{2\sqrt{x-y}} + 2xy^2 = \frac{y'}{2\sqrt{x-y}} - 2x^2yy'$$

$$\frac{1}{2\sqrt{x-y}} + 2xy^2 = y'\left(\frac{1}{2\sqrt{x-y}} - 2x^2y\right)$$

Thus

$$y' = \frac{\frac{1}{2\sqrt{x-y}} + 2xy^2}{\frac{1}{2\sqrt{x-y}} - 2x^2y} = \frac{1 + 4xy^2\sqrt{x-y}}{1 - 4x^2y\sqrt{x-y}}.$$

(l) $y' = \dfrac{y - 2y^{5/2}x^{1/2}}{4x^{3/2}y^{3/2} - x}$

(m) $y' = \dfrac{\cos y + y\sin x}{\cos x + x\sin y}$

76. **(a)** $f'(1) = 2/3$

(b) *Worked Solution:* Differentiating both sides of the equation $g(x) + x\cos(g(x)) = x^3$ gives $g'(x) + \cos(g(x)) - x\sin(g(x))g'(x) = 3x^2$, and setting $x = 0$ gives $g'(0) + \cos(g(0)) = 0$. We can compute $g(0)$ by taking the original equation $g(x)+x\cos(g(x)) = x^3$ and setting $x = 0$ to give $g(0) = 0$, so that $\cos(g(0)) = 1$. It follows that $g'(0) = -1$.

77. **(a)** *Worked Solution:* Differentiating both sides of the equation $y\cos(2x) = x\sin(2y)$ gives

$$y'\cos(2x) - 2y\sin(2x) = \sin(2y) + 2x\cos(2y)y'.$$

Setting $x = \pi/4$ and $y = \pi/2$ gives

$$y'\cos(\pi/2) - \pi\sin(\pi/2) = \sin(\pi) + (\pi/2)\cos(\pi)y'$$

or in other words

$$-\pi = (-\pi/2)y'$$

so that $y' = 2$. Thus the gradient of the tangent line to the curve through the point $(\pi/4, \pi/2)$ is 2, and so the gradient of the tangent is

$$y - \pi/2 = 2(x - \pi/4)$$

or in other words $y = 2x$.

(b) $y = -3x + \pi$

(c) $y = x - 2$

(d) $y = -\sqrt{3}x + 4\sqrt{3}$

(e) $y = \frac{5\sqrt{5}}{8}x - \frac{9}{8}$

78. **(a)** *Worked Solution:* Differentiating both sides of the equation

$$x^2 + 4y^2 = 4$$

and rearranging gives

$$y' = -\frac{x}{4y}.$$

Differentiating this expression then gives

$$y'' = \frac{-4y + 4xy'}{16y^2}.$$

Substituting our expression for y' now gives

$$y'' = \frac{-4y + 4x\frac{-x}{4y}}{16y^2} = \frac{-4y^2 - x^2}{16y^3}.$$

We may now use the original equation to obtain

$$y'' = \frac{-1}{4y^3}.$$

(b) $y'' = \frac{\sqrt{xy} - y}{2x\sqrt{xy}}$

(c) *Worked Solution:* Differentiating the equation

$$x^5 + y^5 = 1$$

gives

$$5x^4 + 5y^4y' = 0$$

which rearranges to give

$$y' = -\frac{x^4}{y^4}.$$

Differentiating this expression gives

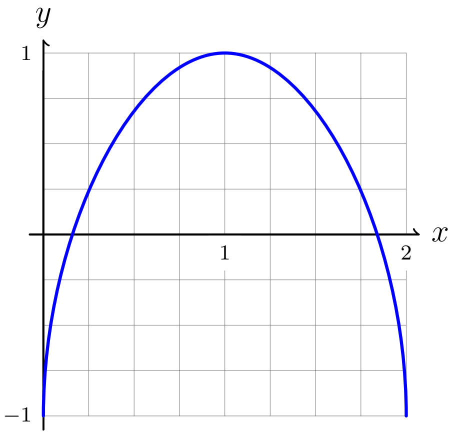
$$\begin{aligned} y'' &= -\frac{y^4 \cdot 4x^3 - x^4 \cdot 4y^3 y'}{y^8} \\ &= -\frac{y^4 \cdot 4x^3 - x^4 \cdot 4y^3 \cdot (-x^4/y^4)}{y^8} \\ &= -\frac{4x^3y^4 + 4x^8y^{-1}}{y^8} \\ &= -4\frac{x^3y^5 + x^8}{y^9} \\ &= -4\frac{x^3}{y^9}(y^5 + x^5). \end{aligned}$$

Now using the original equation we find that

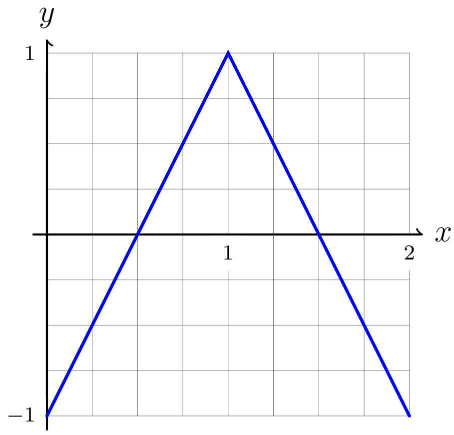
$$y'' = -4\frac{x^3}{y^9}.$$

(d) $y'' = \dfrac{-2xy}{(3y^2 - x)^3}$

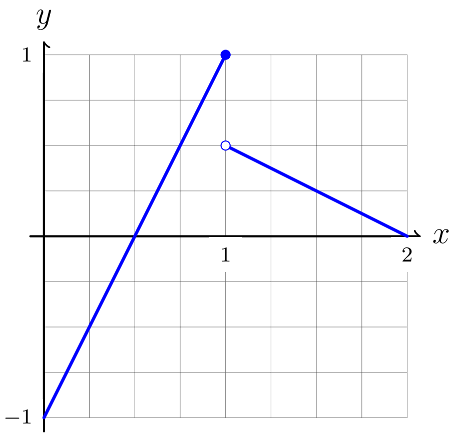
79. **(a)**



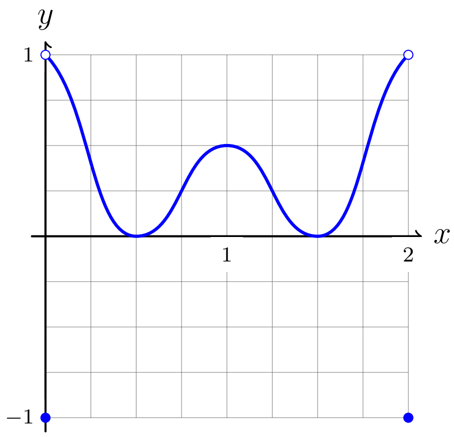
(b)



(c)



(d)



80. **(a)** $-1/4$

(b) 1

(c) *Worked Solution:* The function is continuous and its domain is $\mathbb{R}$.

For $t > 3/4$ we have $4t - 3 > 0$, and so $g(t) = 4t - 3$ and consequently $g'(t) = 4 \neq 0$. So $g'(t)$ exists and is nonzero for all $t > 3/4$, and consequently there are no critical numbers in this range.

For $t < 3/4$ we find, similarly, that there are no critical numbers.

For $t = 3/4$ we find that $g'(t)$ does not exist, since (as can be checked) $\lim_{h\to 0^+} \frac{g(3/4+h)-g(3/4)}{h}$ and $\lim_{h\to 0^-} \frac{g(3/4+h)-g(3/4)}{h}$ exist but are not equal. So $t = 3/4$ is a critical number.

The only critical number is $3/4$.

(d) 0, -2.

(e) *Worked Solution:* The domain of h is $[0, \infty)$. Now $h'(t) = \frac{t^{-3/4}}{4}(6t^{1/2} - 1)$, and in particular it exists for all t in the domain of h, except for $t = 0$. Thus the critical numbers are those t such that $h'(t) = 0$, i.e. $t = 1/36$, and 0.

(f) The critical numbers are 0, where $F'(x)$ does not exist, and 3 and $1/2$, where $F'(x) = 0$.

81. **(a)** The absolute maximum is $f(3) = 9$ and the absolute minimum is $f(1) = 1$.

(b) The absolute maximum is $f(0) = 2$. and the absolute minimum is $f(2) = -14$.

(c) The absolute maximum is $f(2) = 187$ and the absolute minimum is $f(0) = -5$.

(d) *Worked Solution:* We will use the closed interval method. This can be applied because the function f is a product of a polynomial with a square root of a polynomial, and so is continuous on the whole domain $[-1, 3]$.

0. $f'(t) = \sqrt{9-t^2} + \frac{t}{2\sqrt{9-t^2}}(-2t) = \frac{1}{9-t^2}(9 - 2t^2)$. This is defined for all $t \in [-1, 3)$, but not for $t = 3$. So the critical numbers are $t = 3$ and those $t \in [-1, 3)$ for which $f'(t) = 0$, i.e. $3/\sqrt{2}$.

1. The only critical number in the interior $(-1, 3)$ is $3/\sqrt{2}$, and $f(3/\sqrt{2}) = 3/\sqrt{2}\sqrt{9 - 9/2} = 9/2$.

2. $f(-1) = -2\sqrt{2}$ and $f(3) = 0$.

So the absolute maximum and minimum are the largest and smallest numbers from parts 1 and 2. In other words the absolute maximum is $f(3/\sqrt{2}) = 9/2$ and the absolute minimum is $f(-1) = -2\sqrt{2}$.

(e) The absolute maximum is $f(\pi/2) = 3$ and the absolute minimum is $f(7\pi/6) = f(11\pi/6) = -1.5/$

82. **(a)** The only value satisfying the conclusion is $c = 1$.

(b) *Worked Solution:* f is continuous on the interval $[0, 2]$ because $x \geqslant 0$ for all $x \in [0, 2]$, and f is differentiable on this interval because $x > 0$ for all $x \in (0, 2)$. So the hypotheses of the mean value theorem hold.

Next, for the conclusions of the theorem to hold for $c \in (0, 2)$ means that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

Since $f'(x) = \frac{1}{2\sqrt{x}}$, we see that the conclusions hold for c if $\frac{1}{2\sqrt{c}} = \frac{1}{\sqrt{2}}$, so the conclusions hold if and only if $c = \frac{1}{2}$.

83. *Worked Solution:* Let f be the function defined by $f(x) = 2x + \sin(x)$. Then $f(2\pi) = 4\pi + 0 = 4\pi > 0$ and $f(-2\pi) = -4\pi + 0 = -4\pi$. Thus $f(2\pi) > 0 > f(-2\pi)$, and f is continuous on the interval $[-2\pi, 2\pi]$, so by the intermediate value theorem there is $x_0 \in (-2\pi, 2\pi)$ such that $f(x_0) = 0$. In other words the equation has a root x_0.

Suppose that the equation has a second root x_1. Then since $f(x_0) = f(x_1) = 0$ and f is differentiable everywhere, Rolle's theorem shows that there is $c \in (x_0, x_1)$ such that $f'(c) = 0$. However $f'(x) = 2 + \cos(x) \geqslant 2 - 1 = 1 > 0$ so that no such c exists, and so no second root could have existed.

We have shown that there is at least one root, but there cannot be two. So there is exactly one root as required.

84. **(a)** *Worked Solution:* We will show that it is impossible for the equation to have two roots in $[-2, 2]$. Suppose it does, call them x_0 and x_1, and (by swapping them if necessary) assume that $-2 \leqslant x_0 < x_1 \leqslant 2$.

Let f be the function defined by $f(x) = x^3 - 14x + 5$. Then f is a polynomial, so is continuous on $[x_0, x_1]$ and is differentiable on (x_0, x_1). (In fact it is continuous and differentiable everywhere.) Then $f(x_0) = f(x_1) = 0$, so that by Rolle's Theorem there is $c \in (x_0, x_1)$ such that $f'(c) = 0$. Now by computing f' we see that $f'(c) = 3c^2 - 14$. However, since $c \in (-2, 2)$, we have $f'(c) = 3c^2 - 14 < 3 \times 4 - 14 = -2$, so that $f'(c) = 0$ is impossible.

This contradiction means that there could not have been two roots in the first place.

85. **(a)** *Worked Solution:* Since f is differentiable, it satisfies the assumptions of the Mean Value Theorem for the interval $[-1, 2]$. Thus there is $c \in (-1, 2)$ for which

$$\frac{f(2) - f(-1)}{2 - (-1)} = f'(c)$$

or in other words $f(2) - f(-1) = 3f'(c)$. Since $2 \leqslant f'(x) \leqslant 4$, it follows that $6 \leqslant f(2) - f(-1) \leqslant 12$.

86. Define h by $h(x) = g(x) - f(x)$. The given inequalities show that $h(a) = g(a) - f(a) \geqslant 0$ and that $h'(x) = g'(x) - f'(x) \geqslant 0$ for $x \in (a, b)$. And we want to show that $h(b) \geqslant 0$ since then $g(b) - f(b) \geqslant 0$ so that $g(b) \geqslant f(b)$.

Since f and g are continuous on $[a, b]$ and differentiable on (a, b), the same is true for the difference h. So we may apply the Mean Value Theorem: there is $c \in (a, b)$ such that

$$\frac{h(b) - h(a)}{b - a} = h'(c)$$

and consequently $h(b) = (b-a)h'(c) + h(a)$. Since $a < b$ we have $(b-a) > 0$, and since $c \in (a, b)$ we have $h'(c) \geqslant 0$, so that $(b-a)h'(c) \geqslant 0$. And we know that $h(a) \geqslant 0$, so that $(b-a)h'(c) + h(a) \geqslant 0$. Thus $h(b) \geqslant 0$ as required.

87. **(a)** *Worked Solution:* We will use the fact that the local maxima or minima of f are critical points of f.

The function is continuous and its derivative exists for all x, so its critical points are the points where $f'(x) = 0$. Now $f'(x) = 3x^2 - 3$, which is zero for $x = 1$ and $x = -1$. So the possible local maxima and minima occur at $x = \pm 1$.

To determine if they are local maxima or minima, we use the second derivative test, which applies since $f''(x)$ exists and is continuous for all x, and is given by $f''(x) = 6x$.

Since $f''(1) = 6 > 0$, f has a local minimum at 1.

Since $f''(-1) = -6 < 0$, f has a local maximum at -1.

(b) *Worked Solution:* We will use the fact that the local maxima and minima are all critical points. The domain of the function is $\mathbb{R}$ and it is continuous everywhere, so that the critical points are the points where $f'(x) = 0$ or where $f'(x)$ does not exist.

The derivative is $f'(x) = 1 + \frac{1}{3}x^{-2/3}$, which exists for all $x \in \mathbb{R}$ except for $x = 0$. So $x = 0$ is a critical number. There are no other critical numbers, since $f'(x) > 0$ for all $x \neq 0$. So the only possible local maximum or minimum is at $x = 0$. Since $f'(x)$ does not exist there, we cannot use the second derivative test. But the first derivative test applies, and indeed, since $f'(x) > 0$ for all $x \neq 0$, it tells us that 0 is neither a local maximum nor a local minimum.

So f has no local maxima or minima.

(c) f has a local minimum at 0 and a local maximum at -2.

88. **(a)** Increasing on $(-\infty, -2)$ and $(3, \infty)$, decreasing on $(-2, 3)$, local maximum at $(-2, 44)$, local minimum at $(3, -81)$.

(b) *Worked Solution:* First,

$$f'(x) = 12x^2 - 18x + 6 = 6(2x^2 - 3x + 1) = 6(2x-1)(x-1).$$

Thus $f'(x) = 0$ if and only if $x = 1/2$ or $x = 1$. The sign of $f'(x)$ can be computed as follows:

- For $x < 1/2$ we have $(2x-1) < 0$ and $(x-1) < 0$, so that $f'(x) > 0$.
- For $1/2 < x < 1$ we have $(2x-1) > 0$ and $(x-1) < 0$, so that $f'(x) < 0$.
- For $1 < x$ we have $(2x-1) > 0$ and $(x-1) > 0$ so that $f'(x) > 0$.

So f is increasing on $(-\infty, 1/2)$ and $(1, \infty)$ and it is decreasing on $(1/2, 1)$.

As above, the critical numbers of f are $x = 1/2$ and $x = 1$. Now $f(1/2) = 9/4$ and $f(1) = 2$ so that the critical points are $(1/2, 9/4)$ and $(1, 2)$. And we can compute

$$f''(x) = 24x - 18$$

so that $f''(1/2) = -6 < 0$ and $f''(1) = 6 > 0$ and consequently $(1/2, 9/4)$ is a local maximum and $(1, 2)$ is a local minimum.

(c) f is decreasing on $(-\infty, -1)$ and $(0, 1)$, it is increasing on $(-1, 0)$ and $(1, \infty)$, there is a local maximum at $(0, -3)$ and there are local minima at $(-1, -4)$ and $(1, -4)$.

(d) f is increasing on $(-\infty, -\sqrt{2})$ and on $(-\sqrt{2}, 0)$. f is decreasing on $(0, \sqrt{2})$ and on $(\sqrt{2}, \infty)$, and there is a local maximum at $(0, 0)$.

(e) *Worked Solution:* First we compute $f'(x) = 1 - 2\sin(x)$, so that $f'(x) = 0$ when $\sin(x) = 1/2$, or in other words when $x = \pi/6$ and $x = 5\pi/6$. Now by considering the definition of $\sin(x)$ we see that:

- For $0 \leqslant x < \pi/6$ we have $f'(x) > 0$;
- for $\pi/6 < x < 5\pi/6$ we have $f'(x) < 0$;
- for $5\pi/5 < x \leqslant 2\pi$ we have $f'(x) > 0$.

So f is increasing on $[0, \pi/6)$ and $(5\pi/6, 2\pi]$, and it is decreasing on $(\pi/6, 5\pi/6)$.

Next,

$$f''(x) = -2\cos(x),$$

so that $f''(\pi/6) = -2\cos(\pi/6) = -\sqrt{3} < 0$ and $f''(5\pi/6) = -2\cos(5\pi/6) = \sqrt{3} > 0$ so that there is a local maximum at $(\pi/6, \pi/6+\sqrt{3})$ and a local minimum at $(5\pi/6, 5\pi/6-\sqrt{3})$.

(f) f is increasing on $(4, \infty)$ and it is decreasing on $[0, 4)$. It has a local minimum at $(4, -4)$.

89. **(a)** *Worked Solution:* The domain of $f(x) = \frac{1-e^{x^2}}{1-e^{4-x^2}}$ is the set of x such that $1-e^{4-x^2} \neq 0$, or equivalently $e^{4-x^2} \neq 1$, or equivalently $4 - x^2 \neq 0$, or equivalently $x^2 \neq 4$, or equivalently $x \neq \pm 2$. So the domain is

$$\{x \mid x \neq \pm 2\}$$

or, written another way,

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

(b) $\mathbb{R}$

(c) $\{x \mid x \neq k\pi,\ k \in \mathbb{Z}\}$

(d) $(-\infty, 0]$

90. **(a)** 0

(b) $e^s - es^{e-1}$

(c) $(x^4 + 4x^3 - 2x - 2)e^x$

(d) $y' = \frac{d}{dx}(e^{bx^4}) = e^{bx^4}\frac{d}{dx}(bx^4) = e^{bx^4} \cdot 4bx^3 = 4bx^3e^{bx^4}$.

(e) *Worked Solution:*

$$\begin{aligned} y' &= \frac{d}{dt}(e^{2t}\sin(4t)) \\ &= \frac{d}{dt}(e^{2t})\sin(4t) + e^{2t}\frac{d}{dt}\sin(4t) \\ &= 2e^{2t}\sin(4t) + e^{2t} \cdot 4\cos(4t) \\ &= e^{2t}(2\sin(4t) + 4\cos(4t)) \end{aligned}$$

(f) $-e^{x-e^x}$

(g) $\dfrac{(ad-bc)e^{-x}}{(c+de^{-x})^2}$

(h) $\dfrac{-e^{2x}(1+2x)}{2\sqrt{1-xe^{2x}}}$

(i) $4\sin(e^{\cos^2 t})\cos(e^{\cos^2 t})e^{\cos^2 t}\sin t\cos t$

91. The line has equation $y = \dfrac{e}{1-e}(x-1)$.

92. **(a)** The domain of f is $[0,\infty)$ and $f^{-1}(x) = -\ln(1-x^2)$.

(b) *Worked Solution:* The domain of ln is $(0,\infty)$. So the domain of $f(x) = \ln(3-\ln x)$ is the set of x such that $x > 0$ and $3 - \ln x > 0$, or in other words $x > 0$ and $\ln x < 3$, or in other words $x > 0$ and $x < e^3$, or in other words the set $(0, e^3)$.

To find an expression for f^{-1} we first set $y = f(x)$, i.e. $y = \ln(3-\ln x)$. Taking exponentials of both sides, we find $e^y = 3 - \ln x$, so that $\ln x = 3 - e^y$, so that $x = e^{3-e^y}$. Next we interchange x and y to find $y = e^{3-e^x}$, so that

$$f^{-1}(x) = e^{3-e^x}.$$

(c) The domain is $(0,\infty)$, and $f^{-1}(x) = \ln(e^x + 1)$.

93. **(a)** $2 + \ln x$

(b) $-\sin(\ln x)/x$

(c) *Worked Solution:* $f'(x) = \dfrac{d}{dx}(\ln(\cos x)) = \dfrac{1}{\cos x}\dfrac{d}{dx}(\cos x) = \dfrac{1}{\cos x}(-\sin x) = -\tan x$.

(d) $\dfrac{1}{4x(\ln x)^{3/4}}$

(e) *Worked Solution:* Since $\frac{d}{dx}\log_{10}(x) = \frac{1}{x\ln 10}$, the chain rule gives us

$$\frac{d}{dx}\log_{10}(x^2+1) = \frac{1}{(x^2+1)\ln 10}\cdot\frac{d}{dx}(x^2+1) = \frac{1}{(x^2+1)\ln 10}\cdot 2x = \frac{2x}{(x^2+1)\ln 10}.$$

(f) $4x^3 + \ln(4)\cdot 4^x$

(g) $\frac{-1}{x(x^2-1)}$

(h) $\dfrac{-1}{\sqrt{x^2-1}}$

(i) $\dfrac{x}{x-1}$

95. **(a)** One-to-one.

(b) *Worked Solution:* The function is not one-to-one because $f(0) = 1 = f(4)$ but $1 \neq 4$.

(c) Not one-to-one.

(d) *Worked Solution:* The function is one-to-one because if $g(x_1) = g(x_2)$ then $\sqrt[4]{x_1} = \sqrt[4]{x_2}$, and by taking fourth powers we find that $x_1 = x_2$.

(e) Not one-to-one.

(f) *Worked Solution:* The function is one-to-one because no horizontal line crosses its graph more than once.

96. **(a)** $f^{-1}(3) = 1$ and $f^{-1}(-1) = -1$.

(b) $h^{-1}(10) = 8$ and $h^{-1}(2) = 1$.

98. **(a)** $f^{-1}(x) = \frac{4-x}{3}$

(b) $f^{-1}(x) = \frac{x^2-2x}{2}$

(c) $g^{-1}(x) = 1 + \sqrt{1+x}$

(d) *Worked Solution:* First we solve the equation $h(x) = y$ to express x in terms of y. If $\frac{1-2\sqrt{x}}{1+2\sqrt{x}} = y$ then $y + 2y\sqrt{x} = 1 - 2\sqrt{x}$, so that $\sqrt{x}(2y+2) = 1 - y$ and consequently $\sqrt{x} = \frac{1-y}{2y+2}$ so that $x = \left(\frac{1-y}{2y+2}\right)^2$. Next we replace x with y and *vice versa* to obtain $y = \left(\frac{1-x}{2x+2}\right)^2$. Finally, this equation is $y = h^{-1}(x)$, so that

$$h^{-1}(x) = \left(\frac{1-x}{2x+2}\right)^2.$$

(e) *Worked Solution:* First we solve the equation $k(t) = y$ to express t in terms of y. If $y = 4t^2 - 2t$ then $4t^2 - 2t - y = 0$, so by the quadratic formula $t = \dfrac{2 \pm \sqrt{4+16y}}{8} = \dfrac{2 \pm 2\sqrt{1+4y}}{8} = \dfrac{1}{4} \pm \dfrac{1}{4}\sqrt{1+4y}$. Now since $t \geqslant 1/4$, we must have $t = \frac{1}{4} + \frac{1}{4}\sqrt{1+4y}$.

Next, we interchange y and t to obtain $y = \frac{1}{4} + \frac{1}{4}\sqrt{1+4t}$. Finally, this is the equation $y = k^{-1}(t)$, so that

$$k^{-1}(t) = \frac{1}{4} + \frac{1}{4}\sqrt{1+4t}.$$

(f) $f^{-1}(x) = \dfrac{1+x}{2-4x}$.

99. **(a)** *Worked Solution:* We know that $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$. Since $f(0) = 3$, we have $f^{-1}(3) = 0$, so that $(f^{-1})'(3) = \frac{1}{f'(0)}$. Now $f'(x) = 9x^2 + 4x + 8$, so that $f'(0) = 8$, and $(f^{-1})'(3) = 1/8$.

(b) $(f^{-1})'(3) = 1/2$

(c) $(f^{-1})'(4) = 1/2$

(d) $(f^{-1})'(1) = 12/13$

100. Using the quotient rule gives us

$$G'(x) = \frac{f^{-1}(x)\cdot \frac{d}{dx}(f(x)^2) - f(x)^2(f^{-1})'(x)}{f^{-1}(x)^2} = \frac{2f(x)f'(x)f^{-1}(x) - f(x)^2/f'(f^{-1}(x))}{f^{-1}(x)^2}$$

and in particular

$$G'(2) = \frac{2f(2)f'(2)f^{-1}(2) - f(2)^2/f'(f^{-1}(2))}{f^{-1}(2)^2}.$$

Now, since $f(3) = 2$, we have $f^{-1}(2) = 3$, and so

$$G'(2) = \frac{6f(2)f'(2) - f(2)^2/f'(3)}{3^2} = \frac{6\cdot 1\cdot 3 - 1^2/4}{3^2} = 71/36.$$