Remark 2.84. *In this section we will look at inverse functions. Not every function actually has an inverse. The following definition tells us the ones that do have an inverse.*

Definition 2.85 (One-to-one functions). *A function f is called a one-to-one function if it never takes the same value twice. In other words, f is one-to-one if*

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

In terms of graphs, a function is one-to-one if and only if there is no horizontal line that intersects its graph more than once.

Example 2.86. *The function f defined by $f(x) = x^3$ is one-to-one. One way to see this is from the definition: if $x_1 \neq x_2$ then $x_1^3 \neq x_2^3$. Another way to see it is from the graph (which we will not draw here), in which each horizontal line crosses the graph exactly once.*

Example 2.87. *The function g defined by $g(x) = x^2$ is not one-to-one. One way to see this is that, if we let $x_1 = 1$ and $x_2 = -1$, then we have $x_1 \neq x_2$ but $g(x_1) = 1 = g(x_2)$, so that the definition does not hold. Another way to see it is that if $c > 0$, then the horizontal line $y = c$ passes through the graph of $y = g(x)$ twice.*

Definition 2.88 (Inverse functions). *Let f be a one-to-one function with domain A and range B. (Recall that the range of f is the set of all numbers of the form $f(x)$ for some $x \in A$.) Then the inverse function, denoted f^{-1}, has domain B and range A and is defined by the rule*

$$f^{-1}(y) = x \iff y = f(x).$$

Example 2.89. *If f is the function defined by $f(x) = x^3$, then f^{-1} is defined by $f^{-1}(x) = x^{\frac{1}{3}}$. This is because, for this function f^{-1}, the condition*

$$f^{-1}(y) = x$$

is equivalent to

$$y^{\frac{1}{3}} = x$$

which is equivalent to

$$y = x^3,$$

which is equivalent to

$$y = f(x).$$

We will see a perhaps more straightforward way of finding inverse functions shortly.

Here are three important properties of the inverse function. The first is in fact the definition again.

$$f^{-1}(y) = x \iff y = f(x)$$

The next ones are called the *cancellation equations*.

$$f^{-1}(f(x)) = x \qquad \text{for every } x \in A.$$

$$f(f^{-1}(x)) = x \qquad \text{for every } x \in B.$$

How to find the inverse. Let f be a one-to-one function. We find the inverse of f as follows.

1. Write $y = f(x)$.
2. Solve the equation to give x in terms of y, i.e. $x = \cdots$, where the right hand side involves y but not x.
3. Swap x and y, so that you now have an equation of the form $y = \cdots$.
4. The resulting equation is $y = f^{-1}(x)$.

Example 2.90. *Find the inverse of the function f defined by $f(x) = x^3 + 4$.*

Solution We follow the steps above.

1. The equation $y = f(x)$ is $y = x^3 + 4$.
2. Rearranging this gives $y - 4 = x^3$ and then $x = \sqrt[3]{y - 4}$.
3. Swapping x and y gives $y = \sqrt[3]{x - 4}$.
4. So $f^{-1}(x) = \sqrt[3]{x - 4}$.

It is easy to understand the graph of an inverse function in terms of the original function. Indeed:

$$\begin{aligned}(x, y) \text{ lies on the graph of } f &\iff y = f(x) \\ &\iff x = f^{-1}(y) \\ &\iff (y, x) \text{ lies on the graph of } f^{-1}.\end{aligned}$$

This means that the graph of f^{-1} is obtained from the graph of f by switching the x and y values, or in other words, by reflecting in the line $y = x$.

Derivatives of inverse functions. If f is a one-to-one differentiable function and $f'(f^{-1}(a)) \neq 0$, then f^{-1} is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Sketch proof of the formula for $(f^{-1})'(a)$.

One of the two cancellation equations tells us that $f^{-1}(f(x)) = x$ for all x. Differentiating both sides of this equation gives

$$\frac{d}{dx} f^{-1}(f(x)) = \frac{d}{dx}(x)$$

or in other words

$$(f^{-1})'(f(x)) \cdot f'(x) = 1.$$

Rearranging gives

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Now, substituting $f^{-1}(x)$ in place of x, and using the cancellation equation $f(f^{-1}(x)) = x$ gives us

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

□

Example 2.91. *Define f by $f(x) = 2x + \cos(x)$. Find $(f^{-1})'(1)$.*

Solution Working out f' gives $f'(x) = 2 - \sin(x)$. so the formula for $(f^{-1})'(a)$ in the case $a = 1$ gives us

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{2 - \sin(f^{-1}(1))}.$$

It just remains to find $f^{-1}(1)$. However, $f(0) = 2 \cdot 0 + \cos(0) = 0 + 1 = 1$, and this exactly means that $f^{-1}(1) = 0$. So in the end we have

$$(f^{-1})'(1) = \frac{1}{2 - \sin(0)} = \frac{1}{2}.$$

Example 2.92. *Prove the formula for the derivative of $\sqrt[n]{x}$ using the formula for the derivative of the inverse function*

Solution We think of $\sqrt[n]{x}$ as the inverse function of $f(x) = x^n$, where we restrict the domain to $(0, \infty)$ if n is even. Write $g(x) = \sqrt[n]{x}$. Using the formula $f'(x) = nx^{n-1}$ we have

$$g'(x) = 1/f'(g(x)) = \frac{1}{ng(x)^{n-1}} = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}x^{1-\frac{1}{n}}$$

as required.

Example 2.93. *Find the derivatives of the inverse trigonometric functions* $\arcsin$ *and* $\arccos$. *Write them as explicit formulas in* x.

Solution We show here the formula for $\arcsin$. The formula for $\arccos$ is similar. Write $f(x) = \sin(x)$, $g(x) = f^{-1}(x) = \arcsin(x)$. We have

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin(x))}.$$

What is $\cos(\arcsin(x))$? If $\arcsin(x) = y$ then $\sin(y) = x$, and then $\cos^2(y) + \sin^2(y) = 1$ so $\cos^2(y) + x^2 = 1$ or $\cos^2(y) = 1 - x^2$. Since the range of $\arcsin$ is $[-\pi/2, \pi/2]$, we see that $\cos(y)$ is positive, so we get that

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}.$$

We similarly get

$$\arccos'(x) = -\frac{1}{\sqrt{1 - x^2}}$$