2-12 Derivatives of logarithmic functions

Let us now work out the derivative of $\ln$. Let f be the function defined by $f(t) = e^t$. Then it holds that $\ln(f(t)) = t$. Deferentiating and using the chain rule we get:

$$\ln'(f(t))f'(t) = 1$$

$$\ln'(f(t)) = \frac{1}{f'(t)} = \frac{1}{e^t} = \frac{1}{f(t)}.$$

Substituing $f(t) = x$ we get

$$(\ln)'(x) = \frac{1}{x}.$$

So we have:

The derivative of $\ln$.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

The derivative of $\ln(g(x))$.

$$\frac{d}{dx}\left[\ln g(x)\right] = \frac{g'(x)}{g(x)}.$$

Example 2.79. *Find* $\frac{d}{dx} \ln\left(\cos(x)\right)$

Solution Since $\ln(\cos(x)) = \ln g(x)$ where $g(x) = \cos(x)$, we have

$$\frac{d}{dx}\left[\ln(\cos(x))\right] = \frac{\frac{d}{dx}\cos(x)}{\cos(x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x).$$

Example 2.80. *Differentiate the function f defined by $f(x) = \sqrt[3]{\ln(x)}$.*

Solution

$$\frac{d}{dx}\left(\sqrt[3]{\ln(x)}\right) = \frac{d}{dx}\left(\ln(x)\right)^{\frac{1}{3}} = \frac{1}{3}\left(\ln(x)\right)^{\frac{-2}{3}} \cdot \frac{d}{dx}\ln(x) = \frac{1}{3}\left(\ln(x)\right)^{\frac{-2}{3}} \cdot \frac{1}{x} = \frac{1}{3x\ln(x)^{\frac{2}{3}}}$$

Example 2.81. *Find* $\dfrac{d}{dx}\ln\left(\dfrac{x+1}{2x+1}\right)$.

Solution

$$\begin{aligned}
\frac{d}{dx}\ln\left(\frac{x+1}{2x+1}\right) &= \frac{1}{\frac{x+1}{2x+1}}\cdot\frac{d}{dx}\left(\frac{x+1}{2x+1}\right) \\
&= \frac{2x+1}{x+1}\cdot\frac{(2x+1)\cdot\frac{d}{dx}(x+1)-(x+1)\cdot\frac{d}{dx}(2x+1)}{(2x+1)^2} \\
&= \frac{2x+1}{x+1}\cdot\frac{(2x+1)\cdot 1-(x+1)\cdot 2}{(2x+1)^2} \\
&= \frac{2x+1}{x+1}\cdot\frac{-1}{(2x+1)^2} \\
&= \frac{-1}{(x+1)(2x+1)}
\end{aligned}$$

Question What is wrong with the following solution to the above example?
Write first

$$\ln\left(\frac{x+1}{2x+1}\right) = \ln(x+1) - \ln(2x+1).$$

Derive this and get

$$\frac{1}{x+1} - \frac{2}{2x+1}$$

and simplify this to get

$$\frac{-1}{(x+1)(2x+1)}.$$

Solution The problem is that in order to get the equality

$$\ln\left(\frac{x+1}{2x+1}\right) = \ln(x+1) - \ln(2x+1)$$

we need to assume that both $x+1$ and $2x+1$ are positive. So this solution is valid only for $x > \frac{-1}{2}$. The function is defined also when $x < -1$. We can also calculate, separately, in the interval $(-\infty, -1)$, and write there

$$\ln\left(\frac{x+1}{2x+1}\right) = \ln(-x-1) - \ln(-2x-1).$$

We can then derive this and get the same formula for the derivative for $x < -1$.

Example 2.82. *Find the derivative of a^x where $a > 0$ is some number.*

Solution Using the fact that $a = e^{\ln(a)}$ we calculate and we get

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln(a)x} = \ln(a)e^{\ln(a)x} = \ln(a)a^x.$$

Example 2.83. *Find when is the function* $f(x) = \ln x + \frac{1}{x}$ *increasing, decreasing, and when does it have local maximum / minimum.*

Solution We derive first and get

$$f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}.$$

The function is defined for $x > 0$ because that is the domain of definition of $\ln x$. Since x^2 is always positive in the domain of definition, we get that our function increases when $x > 1$ and decreases when $x < 1$. At $x = 1$ we have a local minimum.