Definition 2.77 (The natural logarithm). *The function f defined by $f(x) = e^x$ is increasing, and so is one-to-one. Its domain is $\mathbb{R} = (-\infty, \infty)$, and its range is $(0, \infty)$.*
The natural logarithm, denoted $\ln$, is the inverse function (we will say what we mean by inverse function in the next section). Its domain is $(0, \infty)$, its range is $(-\infty, \infty)$, and it is characterised by the fact that

$$\ln(y) = x \iff y = e^x.$$

Example 2.78.

- $e^0 = 1$, *and so* $\ln(1) = 0$.
- $e^1 = e$, *and so* $\ln(e) = 1$.

The following properties of $\ln$ are all consequences of the definition of $\ln$ together with properties of the exponential function.

Properties of $\ln$.

$$
\begin{aligned}
\ln(ab) &= \ln(a) + \ln(b) \\
\ln(a/b) &= \ln(a) - \ln(b) \\
\ln(a^r) &= r\ln(a) \\
e^{\ln(x)} &= x \\
\ln(e^x) &= x
\end{aligned}
$$