2-9 How derivatives affect the shape of a graph

In this section we will see several more specific ways in which the derivative of a function affects its graph.

The increasing / decreasing test.

- If $f'(x) > 0$ for all x in an open interval I, then f is increasing on I.
- If $f'(x) < 0$ for all x in an open interval I, then f is decreasing on I.

Example 2.69. *On what open intervals is the function f defined by $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ increasing? On what open intervals is it decreasing?*

Solution We first work out the derivative of f.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1).$$

This vanishes for $x = -1, 0, 2$, and we now work out what happens on the intervals obtained by deleting these points.

- On $(-\infty, -1)$ we have $(x+1) < 0$, $x < 0$, $(x-2) < 0$, so that $f'(x) < 0$. Consequently f is decreasing on $(-\infty, -1)$.
- On $(-1, 0)$ we have $(x+1) > 0$, $x < 0$, $(x-2) < 0$, so that $f'(x) > 0$. Consequently f is increasing on $(-1, 0)$.
- On $(0, 2)$ we have $(x+1) > 0$, $x > 0$, $(x-2) < 0$, so that $f'(x) < 0$. Consequently f is decreasing on $(0, 2)$.
- On $(2, \infty)$ we have $(x+1) > 0$, $x > 0$, $(x-2) > 0$, so that $f'(x) > 0$. Consequently f is increasing on $(2, \infty)$.

Now we will see several tests designed to tell whether a critical point is a local maximum or local minimum or neither.

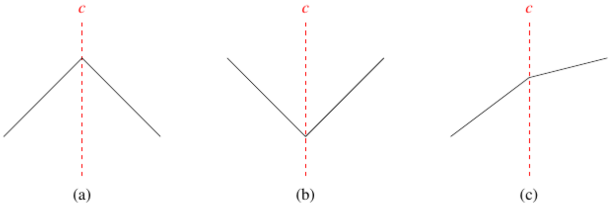
The first derivative test. Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c, then f has neither a local maximum nor a local minimum at c.

The kind of behaviour we are discussing can be depicted as follows.



The rule applies when $f'(x)$ exists for all x close to c but not necessarily equal to c, in particular can apply even when $f'(c)$ does not exist.

The second derivative test. Suppose that c is a critical number of a continuous function f. Suppose that f'' is defined and is continuous near c. Then:

(i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

The test gives no conclusion if $f''(c) = 0$.

Example 2.70. *Classify the local maxima and minima of the function f defined by $f(x) = x^5 - 5x^4$.*

Solution First we find f' and f''.

$$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4), \qquad f''(x) = 20x^3 - 60x^2.$$

So the critical numbers of f are $x = 0$ and $x = 4$.
Now $f''(4) = 20 \cdot 4^4 - 60 \cdot 4^2 = 4160 > 0$, and f'' is defined and continuous near 4, so that the second derivative test applies and tells us that f has a local minimum at 4.
Next, $f''(0) = 0$, so that the second derivative test tells us nothing. We must instead use the first derivative test. Observe that if x is close to 0, i.e. small, then in the expression $f'(x) = 5x^3(x-4)$ the term 5 is positive, $(x-4)$ is negative, and x^3 has the same sign as x. So if x is close to 0 and negative, then $f'(x)$ is positive, and if x is close to 0 and positive, then $f'(x)$ is negative. So $f'(x)$ changes from positive to negative at $x = 0$, and hence f has a local maximum at 0.

Example 2.71. *For the function f of Example 2.17, 0 is a critical number, but neither the first nor the second derivative tests apply. Indeed, f has neither a local maximum nor a local minimum at 0.*