Imagine you are driving a car from point A to point B. The velocity of your car might change throughout the drive: you start slow, then you accelerate, then you slow down et cetera. If the distance between point A and point B is X, and the amount of time the drive took you is T, then the average velocity of your drive was $\frac{X}{T}$. The mean value theorem is the following:

Theorem 2.62 (Mean Value Theorem, imprecise statement). *The velocity of your car at some point, between A and B, was exactly the average velocity $\frac{X}{T}$.*

The precise statement of the mean value theorem is the following:

Theorem 2.63 (The Mean Value Theorem). *Let f be a function satisfying the following hypotheses:*

- f *is continuous on the closed interval* $[a, b]$
- f *is differentiable on the open interval* (a, b).

Then there is a number c *in* (a, b) *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

To prove this theorem, we will begin with the case where $f(a) = f(b)$. This is called Rolle's Theorem:

Theorem 2.64 (Rolle's Theorem). *Let f be a function that satisfies the following three hypotheses:*

- *f is continuous on the closed interval $[a, b]$*
- *f is differentiable on the open interval (a, b).*
- *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof.

We have the following three cases:

Case I : Assume that the function is constant: $f(x) = f(a) = f(b)$ for every x in $[a, b]$. Then the derivative is zero, and we can take c to be any number in (a, b).

Case II : Assume that $f(x) > f(a)$ for some x in (a, b). We know, by the extreme value theorem, that f has a maximum in $[a, b]$. Since $f(x) > f(a) = f(b)$, this maximum is not a nor b. So the maximum is attained at some c in (a, b). But then c is also a local maximum, and it is therefore a crtical number. The derivative $f'(c)$ exists, and by the Theorem of Fermat we have $f'(c) = 0$.

Case III : Assume that $f(x) < f(a)$ for some x in (a, b). This is similar to Case II. Fill in the details!

□

We now use the theorem of Rolle to prove the Mean Value Theorem:

Proof of MVT.

: Write $s = \frac{f(b)-f(a)}{b-a}$. Consider the function $g(x) = f(x) - sx$. We calculate: $g(a) = f(a) - sa$ and $g(b) = f(b) - sb$. This implies:

$$g(a) - g(b) = f(a) - f(b) - sa + sb =$$

$$f(a) - f(b) + s(b-a) =$$

$$f(a) - f(b) + \frac{f(b)-f(a)}{b-a}(b-a) =$$

$$f(a) - f(b) + f(b) - f(a) = 0.$$

This means that $g(a) = g(b)$. The function g satisfies the condition of the theorem of Rolle: Since f and sx are continuous in $[a,b]$ and differentiable in (a,b) the same is true for the difference $g(x) = f(x) - sx$. Therefore, by the theorem of Rolle, there is a point $c \in (a,b)$ such that $g'(c) = 0$. But $g'(c) = f'(c) - s$. This means that

$$f'(c) = s = \frac{f(b)-f(a)}{b-a},$$

which is what we wanted to prove. □

The mean value theorem has the following corollary:

Theorem 2.65. *Assume that $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant on (a, b).*

Proof.

Let c, d be two points in (a, b). We want to show that $f(c) = f(d)$. By the mean value theorem, there is a point e in (c, d) such that $f'(e) = \frac{f(d)-f(c)}{d-c}$. But this means that $f(d) - f(c) = 0$. □

Corollary 2.66. *Assume that* $f'(x) = g'(x)$ *for all* x *in an interval* (a, b)*. Then* $f - g$ *is constant on* (a, b)*.*

Proof.

Consider the function $F(x) = f(x) - g(x)$ and use the above theorem. Fill in the details! $\square$

Example 2.67. *prove that $f(x) = x^3 + x - 1$ has exactly one real root.*

Solution First, notice that $f(0) = -1 < 0$ and $f(1) = 1 > 0$. This means that f has a root between 0 and 1. Next we show that there are no two roots. We do this by contradiction: Assume that a and b are two roots of f, and $a < b$. Then by the theorem of Rolle, there is a point $a < c < b$ such that $f'(c) = 0$. But $f'(x) = 3x^2 + 1$ is always positive, so this is impossible. We thus have only one root.

Example 2.68. *Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all x. How large can $f(2)$ be?*

Solution The MVT gives us that $\frac{f(2)-f(0)}{2-0} = f'(c)$ for some $c \in (0, 2)$. So $f(2) = 2f'(c) - 3 \leq 2 \cdot 5 - 3 = 7$. This means that $f(2)$ is bounded above by 7.