2-5 The Chain Rule

We would like to find the derivative of the composition of two functions $f \circ g$. We will prove the following:

The chain rule. If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Proof.

Write $g(a) = b$. We would like to calculate the derivative of $f \circ g$ at a. If $g(x) \neq g(a)$ for x close enough to a but not equal to a, we calculate:

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} =$$

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} =$$

$$f'(b)g'(a) = f'(g(a))g'(a)$$

and we are done. We have used here the fact that since g is differentiable at a, g is also continuous at a. Therefore, when $x \to a$ it holds that $g(x) \to b$. The problem is that we may have many points in which $g(x) = g(a)$ even when $x \neq a$. For this, we define a new function:

$$Q(y) = \begin{cases} \frac{f(y) - f(b)}{y - b} & \text{if } y \neq b \\ f'(b) & \text{if } y = b \end{cases}$$

Cont.

Notice that Q is a continuous function. It follows that $Q \circ g$ is also a continuous function, since it is the composition of two continuous functions. We define another function:

$$H(x) = \begin{cases} \frac{g(x)-g(a)}{x-a} & \text{if } x \neq a \\ g'(a) & \text{if } x = a \end{cases}$$

Notice that H is defined for all x in the domain of g, and is also continuous. We define a new function F to be $F(x) = (Q \circ g)H$. We claim now that

$$F(x) = ((Q \circ g)H)(x) = \frac{f(g(x)) - f(g(a))}{x - a} \text{ if } x \neq a.$$

Indeed, by using the formula for Q and for H we get that if $g(x) \neq g(a)$ then

$$F(x) = \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} = \frac{f(g(x)) - f(g(a))}{x - a}.$$

If $g(x) = g(a)$ then $H(x) = 0$ and therefore $F(x) = 0$. On the other hand $\frac{f(g(x))-f(g(a))}{x-a} = 0$ because $f(g(x)) = f(g(a))$. The function F is continuous because it is the product of two continuous functoins. We then have that

$$f'(g(x))g'(x) = F(a) = \lim_{x \to a} F(x) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = (f \circ g)'(a)$$

and we are done.

Example 2.38. *Find $F'(x)$ if F is defined by $F(x) = \sqrt{x^2+1}$.*

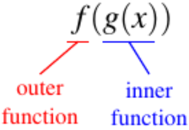
Solution $F = f \circ g$ where $g(x) = x^2 + 1$ and $f(u) = \sqrt{u}$. Thus by the chain rule,

$$F'(x) = f'(g(x)) \cdot g'(x).$$

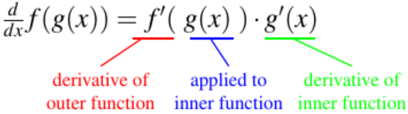
Now, $g'(x) = 2x$ and $f'(u) = \frac{d}{du}\sqrt{u} = \frac{d}{du}u^{\frac{1}{2}} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}}$. Thus

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x) \\ &= f'(x^2+1) \cdot 2x \\ &= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x \\ &= \frac{x}{\sqrt{x^2+1}}. \end{aligned}$$

In applying the chain rule, we think of $f(g(x))$ as an *outer function* f applied to an *inner function* g.



Then the chain rule says:



Example 2.39. *Differentiate* $\sin(x^2)$ *and* $\sin^2(x)$.

Solution Let us differentiate $\sin(x^2)$. In this case we take the outer function to be $\sin(\ \ )$ and the inner function to be x^2. Then the derivative of the outer function is $\cos(\ \ )$, and applying this to the inner function gives $\cos(x^2)$. And the derivative of the inner function is $2x$. So altogether we have

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot 2x = 2x\cos(x^2).$$

Now let us differentiate $\sin^2(x) = [\sin(x)]^2$. In this case we take the outer function to be $[\ \ ]^2$ and the inner function to be $\sin(x)$. So the derivative of the outer function is $2[\ \ ]$ and applying this to the inner function gives $2[\sin(x)]$. And the derivative of the inner function is $\cos(x)$. So altogether we have

$$\frac{d}{dx}\sin^2(x) = \frac{d}{dx}[\sin(x)]^2 = 2[\sin(x)]\cdot\cos(x) = 2\sin(x)\cos(x).$$

The chain rule with the power rule.

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x).$$

Example 2.40. *Let f be the function defined by $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$. Find f'.*

Solution First, we write f as a power of another function.

$$f(x) = \frac{1}{\sqrt[3]{x^2+x+1}} = \frac{1}{(x^2+x+1)^{1/3}} = (x^2+x+1)^{-1/3}.$$

Thus, by the rule above, we have

$$f'(x) = -\frac{1}{3}(x^2+x+1)^{-\frac{1}{3}-1} \cdot (2x+1+0) = -\frac{1}{3}(2x+1)(x^2+x+1)^{-\frac{4}{3}}.$$

Example 2.41. *Differentiate* $y = (2x+1)^5(x^3+x-1)^4$.

Solution Here, y is given to us as the product of the two functions $(2x+1)^5$ and $(x^3+x-1)^4$. So we start by using the product rule. Then on the next line, we use the chain-and-powers rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[(2x+1)^5\right](x^3-x+1)^4 + (2x+1)^5\frac{d}{dx}\left[(x^3-x+1)^4\right] \\ &= 5(2x+1)^4\cdot(2+0)\cdot(x^3-x+1)^4 + (2x+1)^5\cdot 4(x^3-x+1)^3\cdot(3x^2-1+0) \\ &= 2\cdot(2x+1)^4\cdot(x^3-x+1)^3\left[5\cdot(x^3-x+1)+2\cdot(2x+1)\cdot(3x^2-1)\right] \\ &= 2\cdot(2x+1)^4\cdot(x^3-x+1)^3\cdot(17x^3-9x+6x^2+3)\end{aligned}$$

Example 2.42. *Differentiate* $y = \sin(\sin(\sin(x)))$.

Solution We will have to use the chain rule twice, as follows.

$$\begin{aligned}\frac{dy}{dx} &= \cos(\sin(\sin(x))) \cdot \frac{d}{dx}\left[\sin(\sin(x))\right] \\ &= \cos(\sin(\sin(x))) \cdot \left[\cos(\sin(x)) \cdot \frac{d}{dx}\sin(x)\right] \\ &= \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x)\end{aligned}$$