2-6 Implicit differentiation

The functions we have met so far have been described by expressing one variable explicitly in terms of another variable, for example

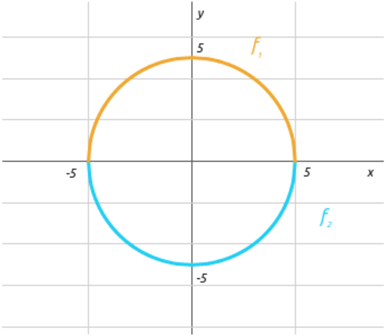
$$y = \sqrt{x^3 + 1} \text{ or } y = x\cos(x).$$

Some functions, however, are described instead by a relation between two variables.
For example,

$$x^2 + y^2 = 25.$$

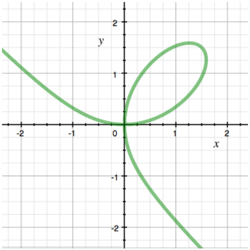
The set of all points (x, y) satisfying this equation determines a curve — in this case it is the circle centered at $(0, 0)$ and with radius 5.

However, this is not the graph of a single function, since there are vertical lines which intersect the curve at two points (and not at one point). However, we say that the equation *implicitly* defines a function f if, when we substitute $f(x)$ in place of y, the equation holds true for all values of x in the domain of f. For example, if we define f_1 and f_2 by $f_1(x) = \sqrt{25 - x^2}$ and $f_2(x) = -\sqrt{25 - x^2}$, then f_1 and f_2 are both implicitly defined by our equation.



Let us take another example. It is called the *folium of Descartes*.

$$x^3 + y^3 = 6xy$$



Here there are many functions implicitly defined by the equation.
Now, even though it is not possible to express y *globally* as a function $f(x)$, it is nevertheless possible to express y *locally* as a function $f(x)$, and to find a formula for $\frac{dy}{dx}$ in terms of x and y.
We have the following theorem, which we shall not prove now. For this theorem we write $F_y(x, y)$ for the function resulting from deriving $F(x, y)$ with respect to y (we think of x as a constant and of y as a variable and derive it accordingly). Similarly, $F_x(x, y)$ is the function resulting from deriving $F(x, y)$ with respect to x.

Theorem 2.43. *Let $F(x, y)$ be a nice function of two variables (we will not say precisely what "nice" means here. All the functions which we will consider here will be nice). Assume that $F(a, b) = 0$ and that $F_y(a, b) \neq 0$. Then there is a continuous differentiable function $f(x)$ such that:*

1. $f(a) = b$.
2. *For x close enough to a and y close enough to b it holds that $F(x, f(x)) = 0$ and moreover $F(x, y) = 0$ exactly when $f(x) = y$.*
3. *The derivative of f is given by $f'(x) = \frac{-F_x(x,y)}{F_y(x,y)}$*

Remark 2.44.

- *This theorem sounds quite complicated at first, but using it is easier. It means that if we look close enough to the point (a, b), the collection of points (x, y) which satisfy the equation $F(x, y) = 0$ look like the graph of a function. Usually we will not be able to write a precise formula for y as a function of x, but that's completely fine.*
- *To find the derivative, we derive $F(x, y)$ with respect to the variable x, where we think of y as a function of x. This will be the same as calculating the quotient $\frac{-F_x(x,y)}{F_y(x,y)}$.*

Example 2.45. *Find an equation of the tangent line to the curve* $x^2 + y^2 = 25$ *at* $(x, y) = (3, 4)$.

Solution Differentiating both sides of $x^2 + y^2 = 25$ gives

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[25\right]$$
$$2x + \frac{d}{dx}y^2 = 0$$
$$2x + 2y\frac{dy}{dx} = 0$$

so that $\frac{dy}{dx} = -\frac{x}{y}$. For $x = 3$, $y = 4$, this gives $\frac{dy}{dx} = -3/4$. So the equation of the tangent is

$$(y - 4) = -\frac{3}{4}(x - 3).$$

What $\frac{dy}{dx}$ means here is that, if f is implicitly defined by our equation, then substituting $y = f(x)$ gives a valid formula for $f'(x)$.

Example 2.46. *Find the points on the folium of Descartes* $x^3 + y^3 = 6xy$ *where* $y' = 0$.

Solution Differentiating both sides of $x^3 + y^3 = 6xy$ gives

$$3x^2 + 3y^2 \cdot \frac{d}{dx}(y) = 6\frac{d}{dx}(x) \cdot y + 6x \cdot \frac{d}{dx}(y)$$

where we have used the chain rule combined with the power law to differentiate y^3, and where we have used the product rule to differentiate $6xy$. This gives

$$3x^2 + 3y^2y' = 6y + 6xy'$$

and rearranging gives

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

This is true whenever $2x \neq y^2$. Thus $y' = 0$ if and only if $y = x^2/2$. However, not all points that satisfy the equation $y = x^2/2$ actually lie on our curve. To find those points, we substitute $y = x^2/2$ into the original equation $x^3 + y^3 = 3xy$ to get

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right).$$

Rearranging this gives us

$$x^3(x^3 - 16) = 0$$

so that $x = 0$ or $x = 2^{4/3}$. The corresponding y values are $y = 0^2/2 = 0$ and $y = (2^{4/3})^2/2 = 2^{5/3}$. But when $y = 0$ it holds that $x = 0$, and the derivative of the equation just gives us $0 = 0$, which contains no information on y'.
So the only required point is

$$(2^{4/3}, 2^{5/3}).$$

Example 2.47. *Find y'' if $x^4 + y^4 = 16$.*

Solution Differentiating both sides of

$$x^4 + y^4 = 16$$

gives

$$4x^3 + 4y^3 y' = 0$$

or equivalently

$$y' = -\frac{x^3}{y^3}.$$

We may now differentiate both sides of this equation to obtain a formula for y'.

$$\begin{aligned} y'' &= \frac{d}{dx}\left[-\frac{x^3}{y^3}\right] \\ &= -\frac{3x^2y^3 - x^3 \cdot 3y^2 \cdot y'}{(y^3)^2} \end{aligned}$$

This formula can be simplified but, more importantly, we can also substitute our known value for y'. This gives:

$$
\begin{aligned}
y'' &= -\frac{3x^2y^3 - x^3 \cdot 3y^2 \cdot \left(-\frac{x^3}{y^3}\right)}{y^6} \\
&= -\frac{3x^2y^4 + 3x^6}{y^7}
\end{aligned}
$$

This is a formula for y'' in terms of just x and y, which is good, but it happens that we can simplify it further, by using the original equation $x^4 + y^4 = 16$.

$$
\begin{aligned}
y'' &= -3x^2\frac{y^4 + x^4}{y^7} \\
&= -3x^2\frac{y^4 + x^4}{y^7} \\
&= -3x^2\frac{16}{y^7} \\
&= -48\frac{x^2}{y^7}
\end{aligned}
$$