2-4 Derivatives of trigonometric functions

In this section we will study trigonometric functions and their derivatives.

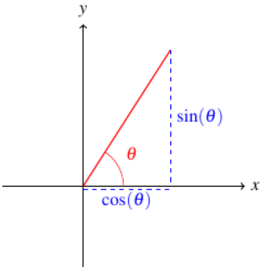
Definition 2.33 (The trigonometric functions). *The trigonometric functions are as follows.*

$$\sin(\theta) \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) \qquad \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Here, $\sin(\theta)$ and $\cos(\theta)$ are defined as follows. Take a line segment of length 1, based at the origin, and making an anticlockwise angle of θ with the positive x-axis. Then $\cos(\theta)$ is defined to be the x-coordinate of the end of the line segment, and $\sin(\theta)$ is defined to be the y-coordinate of the end of the line segment:



In order to differentiate the trigonometric functions, we will need some more facts about them.

Sum-of-angles formulas.

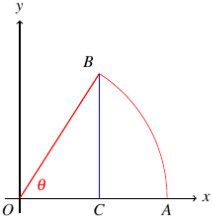
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

Two special limits.

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1 \qquad\qquad \lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}=0$$

The sum of angles formulas should be familiar to you, but the two special limits may not be.

In the next couple of pages we will compute the first of these two limits, namely $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$. The computation of the other one is quite similar. To begin, we assume that $\theta>0$ and we draw a diagram depicting $\sin(\theta)$ and θ as lengths.



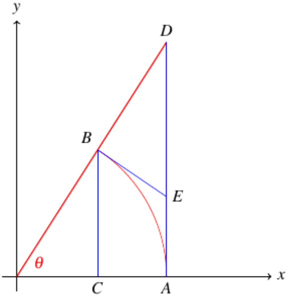
Here, the arc AB is a segment of a circle with radius 1 and making angle θ with the positive x-axis. Since the length of an arc of angle α in a circle of radius r is αr, we have:

$$|AB| = \theta \times 1 = \theta.$$

And by the definition of $\sin(\theta)$ and $\cos(\theta)$ we have:

$$|BC| = \sin(\theta)$$

Now we will extend our diagram to obtain a bit more information.



In this picture we found D by extending OB until its endpoint was directly above A. And we found E by drawing the line segment from B that makes a right-angle with BD, until we meet AD. So by considering the triangle OAD we see that

$$|AD| = \frac{|AD|}{|OA|} = \tan(\theta).$$

Now we write down some inequalities. We clearly have

$$|BC| < |AB|,$$

and since AE and EB form part of a polygon bounding the entire circle, we have

$$|AB| < |AE| + |EB| < |AE| + |ED| = |AD|.$$

So altogether we have

$$|BC| < |AB| < |AD|.$$

Substituting our computations of $|AB|$, $|BC|$ and $|AD|$ into these inequalities gives

$$\sin(\theta) < \theta < \tan(\theta)$$

and a little rearrangement gives

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1.$$

Now, we know that

$$\lim_{\theta \to 0^+} \cos(\theta) = 1 = \lim_{\theta \to 0^+} 1.$$

So by (a one-sided version of) the squeeze theorem, we find that

$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1.$$

Since $\frac{\sin(\theta)}{\theta}$ is even, we know that

$$\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = 1$$

as well, so that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

as required.

Now that we've computed our special limit, we are in a position to work out the following.

Derivatives of $\sin$ and $\cos$.

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$$

$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta)$$

Proof that $\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$.

$$
\begin{aligned}
\frac{d}{d\theta}\sin(\theta) &= \lim_{h\to 0}\frac{\sin(\theta+h)-\sin(\theta)}{h} \\
&= \lim_{h\to 0}\frac{\sin(\theta)\cos(h)+\cos(\theta)\sin(h)-\sin(\theta)}{h} \\
&= \lim_{h\to 0}\left[\sin(\theta)\cdot\frac{\cos(h)-1}{h}+\cos(\theta)\cdot\frac{\sin(h)}{h}\right] \\
&= \sin(\theta)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(\theta)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \sin(\theta)\cdot 0+\cos(\theta)\cdot 1 \\
&= \cos(\theta)
\end{aligned}
$$

$\square$

The proof that $\frac{d}{d\theta}\cos(\theta) = -\sin(\theta)$ is similar.

Further derivatives of trigonometric functions

$$\frac{d}{d\theta}\csc(\theta) = -\csc(\theta)\cot(\theta)$$

$$\frac{d}{d\theta}\sec(\theta) = \sec(\theta)\tan(\theta)$$

$$\frac{d}{d\theta}\tan(\theta) = \sec^2(\theta)$$

$$\frac{d}{d\theta}\cot(\theta) = -\csc^2(\theta)$$

Warning The symbol $\sec^2(\theta)$ means $[\sec(\theta)]^2$, and similarly for $\sin^2(\theta)$, $\cos^2(\theta)$ and so on. On the other hand, $\sin^{-1}(\theta)$ does *not* denote $\frac{1}{\sin(\theta)}$, but instead denots the inverse function, also called $\arcsin(\theta)$, and similarly for the other trigonometric functions.

Example 2.34. *We can derive the differentiation formulas for* tan, sec, csc *and* cot *from the known differentiation formulas for* sin *and* cos*. For example,*

$$\begin{aligned}\frac{d}{d\theta}\cot(\theta) &= \frac{d}{d\theta}\left[\frac{\cos(\theta)}{\sin(\theta)}\right] \\ &= \frac{\sin(\theta)\frac{d}{d\theta}\cos(\theta) - \cos(\theta)\frac{d}{d\theta}\sin(\theta)}{\sin^2(\theta)} \\ &= \frac{-\sin(\theta)\sin(\theta) - \cos(\theta)\cos(\theta)}{\sin^2(\theta)} \\ &= -\frac{\sin^2(\theta)+\cos^2(\theta)}{\sin^2(\theta)} \\ &= -\frac{1}{\sin^2(\theta)} \\ &= -\csc^2(\theta).\end{aligned}$$

Example 2.35. *Find the 57th derivative of* $\cos(x)$*.*

Solution

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= -\sin(x)\\
\frac{d^2}{dx^2}\cos(x) &= \frac{d}{dx}\left(-\sin(x)\right) = -\cos(x)\\
\frac{d^3}{dx^3}\cos(x) &= \frac{d}{dx}\left(-\cos(x)\right) = \sin(x)\\
\frac{d^4}{dx^4}\cos(x) &= \frac{d}{dx}\left(\sin(x)\right) = \cos(x)
\end{aligned}$$

So differentiating $\cos(x)$ four times gives us back $\cos(x)$. That means that the same is true if we differentiate it four times, or eight, or twelve, or ..., or 56 times. (Since $56 = 14 \times 4$.) Thus

$$\frac{d^{56}}{dx^{56}}\cos(x) = \cos(x)$$

and consequently

$$\frac{d^{57}}{dx^{57}}\cos(x) = \frac{d}{dx}\cos(x) = -\sin(x).$$

Example 2.36. *Calculate* $\lim_{x\to 0} \frac{\sin(5x)}{3x}$.

Solution We'll use the fact that $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(5x)}{3x} &= \lim_{x\to 0} \frac{5}{3} \cdot \frac{\sin(5x)}{5x} \\
&= \frac{5}{3} \cdot \lim_{x\to 0} \frac{\sin(5x)}{(5x)} \\
&= \frac{5}{3} \cdot \lim_{t\to 0} \frac{\sin(t)}{t} \\
&= \frac{5}{3} \cdot 1 \\
&= \frac{5}{3}.
\end{aligned}$$

Here, we made a 'substitution' of t in place of $5x$, since if x approaches 0, then so does $5x = t$.

Example 2.37. *Calculate* $\lim_{x\to 0}[x^2 \cot(x)]$.

Solution

$$\begin{aligned}
\lim_{x\to 0}[x^2 \cot(x)] &= \lim_{x\to 0}\left[x^2 \frac{\cos(x)}{\sin(x)}\right] \\
&= \lim_{x\to 0}\left[x \cdot \cos(x) \cdot \frac{x}{\sin(x)}\right] \\
&= \lim_{x\to 0}\left[x \cdot \cos(x) \cdot \left[\frac{\sin(x)}{x}\right]^{-1}\right] \\
&= \lim_{x\to 0}[x] \cdot \lim_{x\to 0}[\cos(x)] \cdot \lim_{x\to 0}\left[\frac{\sin(x)}{x}\right]^{-1} \\
&= 0 \cdot 1 \cdot 1^{-1} \\
&= 0.
\end{aligned}$$