2-3 Differentiation formulas

Derivative of a constant function. Let c be any constant and let f be the function defined by $f(x) = c$. Then $f'(x) = 0$. Or in other words,

$$\frac{d}{dx}c = 0.$$

Derivative of a power function. If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof.

Let f be defined by $f(x) = x^n$. Then we must show that $f'(x) = nx^{n-1}$. And indeed,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \cdots + \binom{n}{r}x^{n-r}h^r + \cdots + h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \cdots + \binom{n}{r}x^{n-r}h^r + \cdots + h^n}{h} \\
&= \lim_{h\to 0} nx^{n-1} + \cdots + \binom{n}{r}x^{n-r}h^{r-1} + \cdots + h^{n-1} \\
&= nx^{n-1}.
\end{aligned}$$

□

Example 2.22.

- *If* $f(x) = x^6$ *then* $f'(x) = 6x^{6-1} = 6x^5$.
- *If* $y = t^4$ *then* $\frac{dy}{dt} = 4t^{4-1} = 4t^3$.
- $\frac{d}{dr}r^3 = 3r^{3-1} = 3r^2$.

Next, we have some rules which tell us how to find the derivatives of new functions from the derivatives of old functions. We will write the proof to some of these rules and give examples.

Derivative of constant multiples. If c is a constant and f is differentiable, then so is cf, and

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)).$$

Example 2.23.

- $\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3 \times 4x^3 = 12x^3$.
- $\frac{d}{dx}(-x) = -\frac{d}{dx}(x) = -\frac{d}{dx}(x^1) = -1 \times 1 \times x^{1-1} = -1$.

The sum rule.

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Proof.

We have

$$\lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} =$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x).$$

This implies that the limit exists and is finite, and therefore $(f+g)'(x) = f'(x)+g'(x)$. □

The difference rule.

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

The sum rule, difference rule, and constant multiple rule can be combined to show that, if f and g are differentiable and a, b are constants, then

$$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x).$$

And indeed, this works when we add together scalar multiples of any number of functions. So:
If $f, g, \ldots, h$ are differentiable and $a, b, \ldots c$ are constants, then

$$\frac{d}{dx}[af(x) + bg(x) + \cdots + ch(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x) + \cdots + c\frac{d}{dx}h(x).$$

We can use this together with the power law to differentiate any polynomial.

Example 2.24. *Compute* $\frac{d}{dx}[3x^2 + 2x + 1]$.

Solution

$$\begin{aligned}\frac{d}{dx}[3x^2 + 2x + 1] &= 3\frac{d}{dx}x^2 + 2\frac{d}{dx}x + \frac{d}{dx}1 \\ &= 3 \cdot 2x^{2-1} + 2x^{1-1} + 0 \\ &= 6x^1 + 2x^0 \\ &= 6x + 2\end{aligned}$$

Example 2.25. *Compute* $\frac{d}{dx}[2x^5 + 4x^3 - 3x^2 + 2]$.

Solution

$$\begin{aligned}\frac{d}{dx}[2x^5 + 4x^3 - 3x^2 + 2] &= 2\frac{d}{dx}x^5 + 4\frac{d}{dx}x^3 - 3\frac{d}{dx}x^2 + \frac{d}{dx}2 \\ &= 2 \cdot 5x^{5-1} + 4 \cdot 3x^{3-1} - 3 \cdot 2x^{2-1} + 0 \\ &= 10x^4 + 12x^2 - 6x\end{aligned}$$

The product rule.

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + \frac{d}{dx}[f(x)]g(x).$$

Or, in other notation,

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x).$$

Proof.

We calculate:

$$\lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h}=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}=$$

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}=$$

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}=$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0}g(x+h)+\lim_{h\to 0}f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=$$

$$f'(x)g(x)+f(x)g'(x).$$

Notice that we have used here the fact that $\lim_{h\to 0}g(x+h)=g(x)$. This follows from the fact that g is continuous at the point x, because g is differentiable at the point x. □

Example 2.26. *Find $F'(x)$ if $F(x) = (4x^3)(7x^4)$.*

Solution Observe that $F(x)$ is the product of the functions given by $4x^3$ and $7x^4$. So by the product rule,

$$\begin{aligned} F'(x) &= \frac{d}{dx}[(4x^3)\cdot(7x^4)] \\ &= (4x^3)\cdot\frac{d}{dx}(7x^4) + \frac{d}{dx}(4x^3)\cdot(7x^4) \\ &= (4x^3)\cdot(28x^3) + (12x^2)\cdot(7x^4) \\ &= 112x^6 + 84x^6. \\ &= 196x^6. \end{aligned}$$

In this case it would have been quicker to first simplify the function and then differentiate. Indeed, $F(x) = (4x^3)(7x^4) = 28x^7$ so that $F'(x) = 28 \times 7x^6 = 196x^6$. However, when we come to use the product rule later we will not be able to simplify in this way.

Example 2.27. *Suppose that f and g are functions and that $f(x) = x^2 \cdot g(x)$. Suppose also that we know that $g(2) = 1$ and $g'(2) = 3$. Find $f'(2)$.*

Solution Even though we don't know what f and g actually are, we can still use the product rule by regarding $f(x)$ as the product of x^2 with $g(x)$:

$$f'(x) = \frac{d}{dx}[x^2 \cdot g(x)] = x^2 \cdot \frac{d}{dx}g(x) + \frac{d}{dx}[x^2] \cdot g(x) = x^2 \cdot g'(x) + 2x \cdot g(x).$$

Now we can substitute $x = 2$ to find that

$$f'(2) = 2^2 \cdot g'(2) + (2 \times 2) \cdot g(2) = 4 \cdot g'(2) + 4 \cdot g(2) = 4 \cdot 3 + 4 \cdot 1 = 16.$$

The quotient rule.

If f and g are differentiable, then if $g(x) \neq 0$ the function $f(x)/g(x)$ is differentiable, and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Or, using briefer notation on the right hand side,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof.

Similar to the product rule, we wll use the fact that $\lim_{h\to 0} g(x+h) = g(x)$. We calculate

$$\lim_{h\to 0} \frac{(f/g)(x+h) - (f/g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} =$$

$$\lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} =$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} \frac{g(x)}{g(x+h)g(x)} +$$

$$\lim_{h\to 0} \frac{f(x)}{g(x)g(x+h)} \lim \frac{g(x) - g(x+h)}{h} =$$

$$\frac{f'(x)g(x)}{g(x)^2} + \frac{-f(x)g'(x)}{g(x)^2} =$$

$$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

$\square$

Example 2.28. *Let* $y = \dfrac{x^2 + 2x - 1}{x^3 - 2}$. *Find* y'.

Solution

$$\begin{aligned}
y' &= \frac{d}{dx}\left[\frac{x^2 + 2x - 1}{x^3 - 2}\right] \\
&= \frac{(x^3 - 2)\frac{d}{dx}(x^2 + 2x - 1) - (x^2 + 2x - 1)\frac{d}{dx}(x^3 - 2)}{(x^3 - 2)^2} \\
&= \frac{(x^3 - 2)(2x + 2) - (x^2 + 2x - 1)(3x^2)}{(x^3 - 2)^2} \\
&= \frac{(2x^4 + 2x^3 - 4x - 4) - (3x^4 + 6x^3 - 3x^2)}{(x^3 - 2)^2} \\
&= \frac{-x^4 - 4x^3 + 3x^2 - 4x - 4}{(x^3 - 2)^2}
\end{aligned}$$

Let's do the same example, but with some advice attached to it:

$$y' = \frac{d}{dx}\left[\frac{x^2+2x-1}{x^3-2}\right]$$

$$= \frac{(x^3-2)\frac{d}{dx}(x^2+2x-1) - (x^2+2x-1)\frac{d}{dx}(x^3-2)}{(x^3-2)^2}$$

always write this out in full

$$= \frac{(x^3-2)(2x+2) - (x^2+2x-1)(3x^2)}{(x^3-2)^2}$$

differentiate *before* multiplying out

$$= \frac{(2x^4+2x^3-4x-4) - (3x^4+6x^3-3x^2)}{(x^3-2)^2}$$

keep the second part in a bracket

$$= \frac{-x^4-4x^3+3x^2-4x-4}{(x^3-2)^2}$$

now subtract

finally, don't expand the bottom

Derivatives of root functions

We recall some relevant notions on root functions. The nth root of a is defined in case n is even and a is non-negative, or in case n is odd and a is any number. When n is odd, the nth root of a is the unique number b which satisfies $b^n = a$. When n is even, the nth root of a is the unique non-negative number b which satifsies $b^n = a$. We write

$$\sqrt[n]{a} = b \text{ or } a^{\frac{1}{n}} = b.$$

For a rational number (that is: a number which we can write as a quotient of two integers $\frac{m}{n}$ we write

$$a^{\frac{m}{n}} = \sqrt[n]{a}^m.$$

The usual rules for manipulating powers hold here:

Rules for manipulating powers.

- $\dfrac{1}{a^n} = a^{-n}$
- $a^p \cdot a^q = a^{p+q}$
- $a^{1/p} = \sqrt[p]{a}$
- $a^p b^p = (ab)^p$
- $\dfrac{a^p}{a^q} = a^{p-q}$
- $(a^p)^q = a^{pq}$

The function $f(x) = x^{\frac{m}{n}}$ is differentiable for $x \neq 0$, and it is also differentiable at $x = 0$ if $\frac{m}{n} \geq 1$. We have the following rule, which generalizes the previous rule for the derivative of x^n:

General power rule.

$$\frac{d}{dx}(x^{\frac{m}{n}}) = \frac{m}{n} x^{\frac{m}{n}-1}.$$

Or: if we just write $\frac{m}{n} = r$ then:

$$\frac{d}{dx}(x^r) = rx^{r-1}.$$

Remark 2.29.

- *Notice that this can also be used for negative r: if $r < 0$, then $x^r = \frac{1}{x^{-r}}$, and the rules of derivatives work the same for negative powers (and the function is differentiable for $x \neq 0$ in case the denominator in r is even, and for $x > 0$ in case the denominator in r is odd).*
- *This rule of derivation works the same in case the exponent r is not a rational number, but a real number. We will explain later what we mean by expression such as $2^{\sqrt{2}}$ or 3^{π}.*

Example 2.30.

- *If* $y = \frac{1}{x}$, *then* $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2}$.
- $\frac{d}{dt}\left[\frac{6}{t^3}\right] = 6\frac{d}{dt}\left[\frac{1}{t^3}\right] = 6\frac{d}{dt}[t^{-3}] = 6 \times (-3) \times t^{-3-1} = -18t^{-4}$.
- *If* $f(x) = x^{0.8}$, *then* $f'(x) = 0.8x^{-0.2}$.
- *If* $y = \frac{1}{\sqrt[3]{x^2}}$, *then*

$$y = \frac{1}{(x^2)^{1/3}} = \frac{1}{x^{2/3}} = x^{-\frac{2}{3}}$$

so

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-\frac{2}{3}}) = -\frac{2}{3}x^{-\frac{2}{3}-1} = -\frac{2}{3}x^{-\frac{5}{3}}.$$

Example 2.31. *Differentiate the function f defined by $f(t) = \sqrt{t}(a + bt)$.*

Solution

$$
\begin{aligned}
f'(t) &= \frac{d}{dt} f(t) \\
&= \frac{d}{dt}\left(\sqrt{t}(a + bt)\right) \\
&= \frac{d}{dt}\left(t^{\frac{1}{2}}(a + bt)\right) \\
&= \frac{d}{dt}\left(at^{\frac{1}{2}} + bt^{\frac{3}{2}}\right) \\
&= a\frac{d}{dt}\left(t^{\frac{1}{2}}\right) + b\frac{d}{dt}\left(t^{\frac{3}{2}}\right) \\
&= \frac{1}{2}at^{-\frac{1}{2}} + \frac{3}{2}bt^{\frac{1}{2}}.
\end{aligned}
$$

Example 2.32. *Differentiate* $y = \dfrac{\sqrt{x}}{1+x^2}$.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1+x^2)\frac{d}{dx}\left(\sqrt{x}\right) - \sqrt{x}\frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\
&= \frac{(1+x^2)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - x^{\frac{1}{2}}(2x)}{(1+x^2)^2} \\
&= \frac{\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{3}{2}} - 2x^{\frac{3}{2}}}{(1+x^2)^2} \\
&= \frac{\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{3}{2}}}{(1+x^2)^2}
\end{aligned}$$