Definition 2.9 (The derivative). *Let f be a function. The derivative of f, denoted f', is the function defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of f' is

$$\operatorname{dom}(f') = \left\{ x \;\middle|\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists and is finite} \right\}.$$

Example 2.10. *Let f be the function defined by $f(x) = x^3 - x$. Use the definition of the derivative to find a formula for $f'(x)$.*

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 1) \\
&= 3x^2 - 1.
\end{aligned}
$$

Example 2.11. *Let $f(x) = \sqrt{x}$. Use the definition of the derivative to find a formula for $f'(x)$.*

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

The domain of $f'(x)$ is $(0, \infty)$.

Example 2.12. *Let f be the function defined by $f(x) = |x|$. What is the domain of f'?*

Solution We already saw in Example 2.8 that $f'(0)$ does not exist. We will show here that $f'(x)$ *does* exist if $x \neq 0$, so that
$\mathrm{dom}(f) = \{x \mid f'(x) \text{ exists}\} = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.
Case 1: If $x > 0$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h} \\ &= \lim_{h \to 0} \frac{h}{h} \\ &= \lim_{h \to 0} 1 \\ &= 1 \end{aligned}$$

and in particular $f'(x)$ does indeed exist. Here, we were able to replace $|x|$ with x since $x > 0$. And we were able to replace $|x+h|$ with $(x+h)$ for the following reason: we know that $x > 0$, and since we are looking at a limit as h approaches 0, we can assume that h is small — small enough that $(x+h) > 0$ also.

Case 2: If $x < 0$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} \\ &= \lim_{h\to 0} (-1) \\ &= -1 \end{aligned}$$

and in particular $f'(x)$ does indeed exist. Here, we were able to replace $|x|$ with $-x$ since $x < 0$. And we were able to replace $|x+h|$ with $-(x+h)$ for the following reason: we know that $x < 0$, and since we are looking at a limit as h approaches 0, we can assume that h is small — small enough that $(x+h) < 0$ also.

So we have confirmed that $f'(x)$ exists when $x \neq 0$, and in fact we have the following formula for f'.

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Example 2.13. *Let* k *be the function defined by* $k(x) = \sqrt[3]{x}$*. Show that* $k'(0)$ *does not exist.*

Solution $k'(0)$, if it exists, is given by the following limit.

$$k'(0) = \lim_{h\to 0} \frac{k(0+h)-k(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}-\sqrt[3]{0}}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0} \frac{1}{h^{\frac{2}{3}}}$$

But this limit is not finite. Therefore, the derivative does not exist at the point 0.

Example 2.14. *Let H be the Heaviside function defined by*

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

Show that $H'(0)$ does not exist.

Solution We must show that

$$\lim_{h \to 0} \frac{H(h) - H(0)}{h}$$

does not exist. Now, $H(0) = 1$, but the value of $H(h)$ will depend on whether $h > 0$ or $h < 0$. So to understand this limit we will look at the right and left handed limits. The right-handed limit does exist:

$$\lim_{h \to 0^+} \frac{H(h) - H(0)}{h} = \lim_{h \to 0^+} \frac{1 - 1}{h} = \lim_{h \to 0^+} \frac{0}{h} = \lim_{h \to 0^+} 0 = 0.$$

But the left-handed limit does not:

$$\lim_{h \to 0^-} \frac{H(h) - H(0)}{h} = \lim_{h \to 0^-} \frac{0 - 1}{h} = \lim_{h \to 0^-} \frac{-1}{h}$$

So the two-sided limit $\lim_{h \to 0} \frac{H(h)-H(0)}{h}$ does not exist either, and consequently $H'(0)$ does not exist. In fact, the underlying reason that $H'(0)$ does not exist is that H is not continuous at 0, as in the following theorem.

Theorem 2.15. *If f is differentiable at a, i.e. if $f'(a)$ exists, then f is continuous at a. Phrased differently, if f is not continuous at a then f cannot be differentiable at a.*

Proof.

Assume that f is differentiable at a. We calculate:

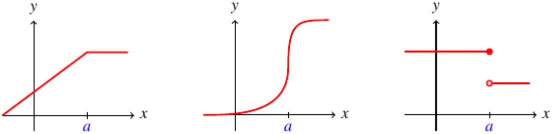
$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a}(x-a)$$

$$= f'(a)\cdot 0 = 0.$$

This shows us that $\lim_{x\to a} f(x) = f(a)$, and so f is continuous at a. Notice that in order to write the limit of the product as product of the limits, we have used the fact that both limits exists and are finite. □

Three ways a function can fail to be differentiable

Here are three graphs depicting how a function can fail to be differentiable at a number a.



(a) corner (b) "infinite slope" (c) discontinuity

Example 2.16. *Let f be the function defined as follows:*

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

Then $f'(x)$ exists for all x, and is given by the following formula:

$$f(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Or in other words, $f(x) = 2|x|$. We have seen that the absolute value function is continuous but not differentiable in zero. This shows us that $f'(x)$ does not have to be differentiable everywhere, even if f is.

The next example shows that the derivative does not even have to be continuous:

Example 2.17. *Let f be the function defined as follows.*

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then $f'(x)$ exists for all x, and is given by the following formula:

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(The formula for $f'(x)$ when $x \neq 0$ is an exercise in differentiation rules, which we will see later. The formula for $f'(0)$ requires you to use the precise definition of the derivative together with the squeeze theorem.) Now, observe that f' is not continuous at $a = 0$. So we see that a function may be differentiable everywhere, but that its derivative may not be continuous.

Definition 2.18 (Leibniz notation). *If we use the traditional notation $y = f(x)$ to indicate that the variable y depends on the variable x by means of the function f, then there are many different ways to denote the derivative, as follows.*

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$$

The notation involving $\frac{d}{dx}$ is called Leibniz notation. We will switch between notations frequently.

Definition 2.19 (Higher derivatives). *Let f be a function. Its derivative f' is another function. That means that we can differentiate f' to produce another function, $(f')'$, which is called the second derivative and denoted f''. Differentiating once more gives $(f'')'$, which is denoted f''' and called the third derivative. Repeating the process, we can define the nth derivative of f, which is denoted by $f^{(n)}$.*
In Leibniz notation, if $y = f(x)$, then we would write

$$f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = \frac{d^n f}{dx^n} = \frac{d^n}{dx^n} f(x).$$

Remark 2.20. *As was mentioned before, if $f(t)$ is a function which described the location of an object at time t, then the derivative $f'(t)$ gives us the velocity of the object at time t. The second derivative $f''(t)$ gives us the acceleration of the object at time t, since the acceleration measures the rate of change of the velocity with respect to the time.*

Example 2.21. *If f is defined by $f(x) = x^3 - x$, then find f' and f'' and f'''.*

Solution First,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1 \\
&= 3x^2 - 1.
\end{aligned}$$

Next,

$$
\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 1 - 3x^2 + 1}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h\to 0} 6x + 3h \\
&= 6x.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
f'''(x) &= \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} \\
&= \lim_{h\to 0} \frac{6(x+h) - 6x}{h} \\
&= \lim_{h\to 0} \frac{6h}{h} \\
&= \lim_{h\to 0} 6 \\
&= 6.
\end{aligned}
$$