2. Derivatives and rates of change

Tangents

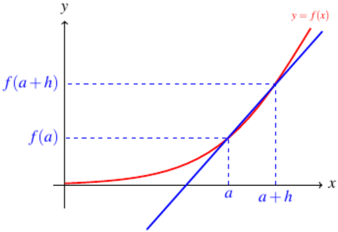
Definition 2.1. *The tangent line to the curve* $y = f(x)$ *at the point* $(a, f(a))$ *is the line through* $(a, f(a))$ *with gradient*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists. Notice that this limit is the same as

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This is pictured in the graph below, which shows that $\frac{f(a+h)-f(a)}{h}$ *is the gradient of the line that crosses* $y = f(x)$ *at* $(a, f(a))$ *and* $(a+h, f(a+h))$.



Example 2.2. *Find the equation of the tangent line to the curve* $y = x^2$ *through the point* $(1, 1)$.

Solution Let f be the function defined by $f(x) = x^2$, so that our curve is given by $y = f(x)$. Then the gradient of the line is

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 2h + 1 - 1}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 2h}{h} \\ &= \lim_{h \to 0} (h + 2) \\ &= 2 \end{aligned}$$

And so the equation of the tangent line is $(y - 1) = 2(x - 1)$, or in other words $y = 2x - 1$.

Velocities

Suppose that an object is moving along a line according to the equation $s = f(t)$ where s is the *displacement*, i.e. position along the line, t is tthe time, and $f(t)$ is the *position function*. The *average velocity* of the object between times a and $a + h$ is then

$$\text{average velocity} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

And the *instantaneous velocity* at time a is

$$\text{instantaneous velocity} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This is the gradient of the graph of $y = f(x)$ at $(a, f(a))$.

Derivatives

Definition 2.3 (The derivative of f at a). *The derivative of a function f at a number a, denoted by $f'(a)$, is defined by*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists and is finite. If $f'(a)$ exists, then we say that f is differentiable at a.

Example 2.4. *Let f be the function defined by $f(x) = x^2 - 8x + 9$. Using the definition of the derivative, find the derivative of f at a.*

Solution We start the question by simply writing out the definition of the derivative.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Next, we write out $f(a+h)$ and $f(a)$ using the definition of f. Remember, when you write down $f(a)$, you do it by taking the definition of $f(x)$ and replacing every x with a. And when you write down $f(a+h)$, do it by replacing every x with $(a+h)$ — remember to include the brackets, as it will save you from making a lot of mistakes.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \end{aligned}$$

And now we expand, simplify, and try to work out the limit.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\
&= \lim_{h\to 0} \frac{[a^2 + 2ah + h^2 - 8a - 8h + 9] - [a^2 - 8a + 9]}{h} \\
&= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
&= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} \\
&= \lim_{h\to 0} (2a + h - 8) \\
&= 2a - 8.
\end{aligned}$$

So $f'(a) = 2a - 8$.

Example 2.5. *Let f be the function defined by $f(x) = 2x^2 + x - 3$. Find $f'(2)$.*

Solution We begin, as always, by writing out the definition of $f'(2)$. This is of course just the same as the definition of $f'(a)$, but with 2 substituted in place of a.

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{[2(2+h)^2 + (2+h) - 3] - [2\cdot 2^2 + 2 - 3]}{h} \\
&= \lim_{h\to 0} \frac{[2\cdot 2^2 + 8h + 2h^2 + 2 + h - 3] - [2\cdot 2^2 + 2 - 3]}{h} \\
&= \lim_{h\to 0} \frac{2\cdot 2^2 + 8h + 2h^2 + 2 + h - 3 - 2\cdot 2^2 - 2 + 3}{h} \\
&= \lim_{h\to 0} \frac{9h + 2h^2}{h} \\
&= \lim_{h\to 0} (9 + 2h) \\
&= 9.
\end{aligned}$$

Here are some important points to note when you are answering a question like this.

- Always start by writing out the definition, e.g.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{or} \qquad f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}.$$

- Be careful when writing out $f(a+h)$. Take the definition of $f(x)$ and put $(a+h)$ in place of every x. Include the brackets! You will avoid mistakes that way.
- Make sure that you include the $\lim_{h\to 0}$ in every step, until you reach a point where you can actually compute the limit. (In the examples above, we had $\lim_{h\to 0}$ on every line until the very last one.)
- If the question asks you to work out $f'(2)$, then do that! Don't work out $f'(a)$ for a general a first. (There's probably a reason why the question is written that way. We will see examples where in some special values the calculation of the derivative is different than for other values).

Observe that $f'(a)$ is the gradient of the tangent line to $y = f(x)$ at $(a, f(a))$. Observe also that if we regard $f(t)$ as a position, then $f'(a)$ is the instantaneous velocity at time a.

Example 2.6. *Let g be the function defined by $g(x) = \frac{1}{x+2}$. Use the definition of the derivative to find a formula for $g'(a)$.*

Solution

$$\begin{aligned}
g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[g(a+h)-g(a)\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(a+h)+2} - \frac{1}{a+2}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(a+2)-(a+h+2)}{(a+h+2)(a+2)}\right] \\
&= \lim_{h\to 0} \left[\frac{-1}{(a+h+2)(a+2)}\right] \\
&= \frac{-1}{(a+0+2)(a+2)} \\
&= -\frac{1}{(a+2)^2}
\end{aligned}$$

Example 2.7. *Let f be the function defined by*

$$f(x) = \begin{cases} x^3 & \textit{if } x \geqslant 0 \\ -x^3 & \textit{if } x < 0. \end{cases}$$

Show that $f'(0) = 0$.

Solution Let us start our working out as usual.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h)}{h} \end{aligned}$$

We would now like to substitute in the definition of $f(h)$ and then work out the limit, but the formula for $f(h)$ depends on whether $h \geqslant 0$ or $h < 0$, and when we are working out the limit we do not know which of these applies. However, we can easily work out the left and right handed limits, as follows.

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0^+} \frac{f(h)}{h}$$
$$= \lim_{h\to 0^+} \frac{h^3}{h}$$
$$= \lim_{h\to 0^+} h^2$$
$$= 0.$$

Here, we were able to replace $f(h)$ with h^3 since it is a limit as h approaches 0 from the right, so that we know $h > 0$ and consequently $f(h) = h^3$. And now we do the left-handed limit.

$$
\begin{aligned}
&\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0^-} \frac{f(h)}{h} \\
&= \lim_{h\to 0^-} \frac{-h^3}{h} \\
&= \lim_{h\to 0^-} (-h^2) \\
&= 0.
\end{aligned}
$$

Again, since this is a limit as h approaches 0 from the left, we knew that $h < 0$, and so were able to replace $f(h)$ with $-h^3$. Now, since

$$
\lim_{h\to 0^+} \frac{f(h)}{h} = 0 = \lim_{h\to 0^-} \frac{f(h)}{h}
$$

we can conclude that

$$
f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = 0
$$

as required.

Example 2.8. *Define f by $f(x) = |x|$. Does $f'(0)$ exist?*

Solution Recall the definition of the absolute value:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

We start working out the derivative as follows.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

Now we see that, since the definition of $|h|$ depends on whether h is positive or negative, we must examine the left and right handed limit separately. This gives us

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Here we were able to replace $|h|$ with h since we are looking at a limit as h approaches 0 *from the right*, so that $h > 0$ and consequently $|h| = h$. And

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

Here we were able to replace $|h|$ with $-h$ since were looking at a limit as h approaches 0 *from the left*, so that $h < 0$ and consequently $|h| = -h$. Since $\lim_{h \to 0^+} \frac{|h|}{h}$ and $\lim_{h \to 0^-} \frac{|h|}{h}$ are not equal, it follows that $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. Consequently $f'(0)$ does not exist.