University of Aberdeen

Advanced Mathematics I-1
Answers and Worked Solutions

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1. **(a)** -1.25.

(b) f has domain $[-2, 4]$ and range $[-2, 3]$.

(c) g has domain $[-3, 4]$ and range $[-0.25, 4]$. (Here -0.25 is an estimate.)

2. **(a)** $\{x \mid x \neq -2, 2\}$, or in other words $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

(b) $\mathbb{R}$.

(c) $\{x \mid x \neq 1, -2\}$, or in other words $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$

(d) *Worked Solution:* $\sqrt{2-t}$ is defined when $2 - t \geqslant 0$, or in other words when $t \leqslant 2$. $\sqrt{2+t}$ is defined when $2 + t \geqslant 0$, or in other words when $t \geqslant 2$. So $\sqrt{2-t} + \sqrt{2+t}$ is defined when both condition s$t \geqslant -2$ and $t \leqslant 2$ are true. Thus the range is $[-2, 2]$.

(e) $(-\infty, 0) \cup (6, \infty)$.

(f) *Worked Solution:* $\dfrac{1}{\sqrt[3]{u^2 - 6u}}$ is defined when $\sqrt[3]{u^2 - 6u}$ is defined and nonzero. Now $\sqrt[3]{u^2 - 6u}$ is always defined, and it is nonzero when $u^2 - 6u$ is nonzero, or in other words when $u \neq 0, 6$. Thus the domain is $\{u \mid u \neq 0, 6\}$, or in other words $(-\infty, 0) \cup (0, 6) \cup (6, \infty)$.

(g) *Worked Solution:* For the function to be defined we require that both square roots exist, or in other words that

$$1 + t \geqslant 0$$

for $\sqrt{1+t}$ to exist, and that

$$4 - \sqrt{1+t} \geqslant 0$$

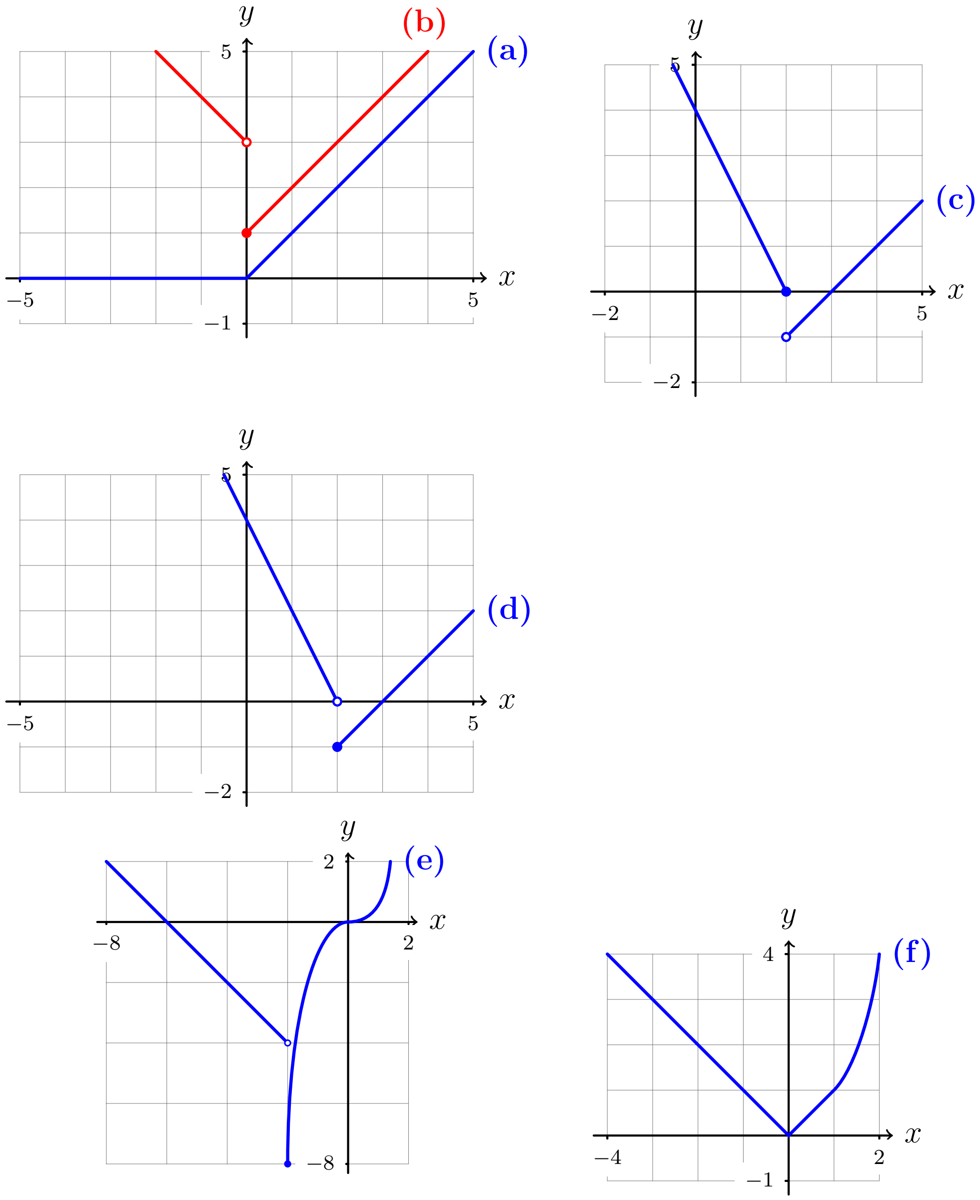
for $\sqrt{4 - \sqrt{1+t}}$ to exist. The first of these states that $t \geqslant -1$, and the second is equivalent to $4 \geqslant \sqrt{1+t}$, which is equivalent to $16 \geqslant 1 + t$, which is equivalent to $t \leqslant 15$. So altogether the requirement is that

$$-1 \leqslant t \leqslant 15.$$

So the domain is $[-1, 15]$.

(h) The domain is $[-1, 0) \cup (0, 15]$.

3.



4. **(a)** $f(x) = \begin{cases} 1-x, & 0 \leqslant x \leqslant 1 \\ x-1, & 1 < x \leqslant 3 \end{cases}$ and the range is $[0, 2]$.

(b) $f(x) = \begin{cases} x+2, & -2 \leqslant x < -1 \\ -x, & -1 \leqslant x < 1 \\ x-2, & 1 \leqslant x \leqslant 2 \end{cases}$ and the range is $[-1, 1]$.

(c) *Worked Solution:* In the range $-2 \leqslant x \leqslant 0$ the graph is the line with gradient $3/2$ and y-intercept 3, so the function is given by $\frac{3}{2}x + 3$ in this range. And in the range $0 \leqslant x \leqslant 2$ the graph is the line with gradient $-3/2$ and y-intercept 3, so the function is given by $3 - \frac{3}{2}x$ in this range. So we have the following:

$$f(x) = \begin{cases} \frac{3}{2}x + 3, & -2 \leqslant x < 0 \\ 3 - \frac{3}{2}x, & 0 \leqslant x \leqslant 2 \end{cases}$$

We have chosen $x < 0$ in the first line to avoid defining $f(x)$ twice. But we could just as well have written the following, and the answer would still be right.

$$f(x) = \begin{cases} \frac{3}{2}x + 3, & -2 \leqslant x < 0 \\ 3 - \frac{3}{2}x, & 0 \leqslant x \leqslant 2 \end{cases}$$

By looking at which y-values are attained by the function we see that the range is $[0, 3]$.

(d) $f(x) = \begin{cases} -2, & -2 \leqslant x < 0 \\ 2, & 0 \leqslant x < 2 \end{cases}$ and the range is $[-2, 2)$.

5. **(a)** The function is odd. It is not even.

(b) The function is neither odd nor even.

(c) The function is even. It is not odd.

(d) The function is even. It is not odd.

(e) The function is both odd and even.

6. **(a)** The function is even. It is not odd.

(b) The function is odd. It is not even.

(c) The function is neither even nor odd.

(d) *Worked Solution:* $f(x) = \dfrac{x^2}{x^3 + x}$ so that

$$f(-x) = \frac{(-x)^2}{(-x)^3 - x} = \frac{x^2}{-x^3 - x} = -\frac{x^2}{x^3 + x} = -f(x)$$

and consequently f is odd. Since $f(-x) \neq f(x)$, is it not even.

(e) The function is even. It is not odd.

(f) The function is neither even nor odd.

(g) The function is odd. It is not even.

(h) *Worked Solution:* $g(x) = |x| \cdot x^2$ so that

$$g(-x) = |-x| \cdot (-x)^2.$$

Now $|-x| = |x|$ and $(-x)^2 = (-x)(-x) = x^2$, so that

$$g(-x) = |-x| \cdot (-x)^2 = |x| \cdot x^2 = g(x).$$

Thus g is even. Since $g(x) \neq -g(x)$, it is not odd.

(i) The function is odd. It is not even.

(j) *Worked Solution:* We work out $(p+q)(-x)$ and see whether it is equal to $(p+q)(x)$ or $-(p+q)(x)$. We find that

$$(p+q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p+q)(x)$$

so that $p+q$ is even. It is not (necessarily) odd.

(k) The function is odd. It is not (necessarily) even.

(l) The function is odd.

(m) *Worked Solution:* Since q is even and r is odd we have

$$(q \circ r)(-x) = q(r(-x)) = q(-r(x)) = q(r(x))$$

so that $q \circ r$ is even.

(n) The function is even.

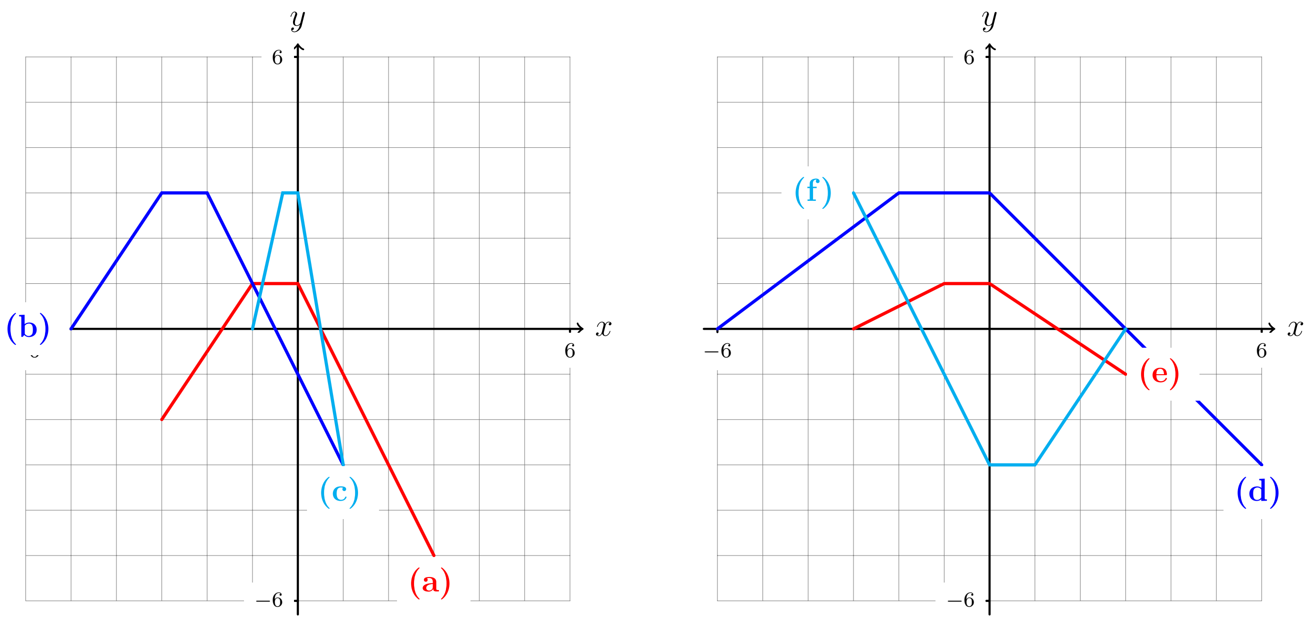
(o) The function is odd.

7. *Worked Solution for g:* Recall that $g(x) = \frac{f(x)+f(-x)}{2}$, so that

$$g(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(-x)+f(x)}{2} = \frac{f(x)+f(-x)}{2} = g(x),$$

and consequently g is even.

8.



9. **(a)** $g(x) = f(x-2) - 1$.

(b) *Worked Solution:* The graph of $h(x)$ is obtained from the graph of f by the following steps: First, reflect the graph of f in the x-axis. Second, move it 2 to the right. Third, move it 1 down. After the first step, we have the graph of the function $-f(x)$. After the second step, we have the graph of the function $-f(x-2)$. After the third step, we have the graph of the function $-f(x-2)-1$. So $h(x) = -f(x-2) - 1$.

(c) $k(x) = -\frac{1}{3}f(-x-2)$.

10. **(a)** $(f+g)(x) = x^3 + x^2 - 2$, and its domain is $\mathbb{R}$.

(b) $(f-g)(x) = x^3 - x^2 + 6$, and its domain is $\mathbb{R}$.

(c) $(fg)(x) = (x^3+2)(x^2-4)$, and its domain is $\mathbb{R}$.

(d) *Worked Solution:*

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x^3+2}{x^2-4}$$

Its domain is the collection of all points x that are in the domain of f, and the domain of g, and that satisfy $g(x) \neq 0$. The domain of f and g are both $\mathbb{R}$. And $g(x) = 0$ if and only if $x = 2$ or $x = -2$. So the domain of f/g is $\{x \mid x \neq 2, -2\}$ or in other words $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

11. **(a)** *Worked Solution:* $(f+g)(x) = f(x) + g(x) = \sqrt{5-x} + \sqrt{x^2-4}$. The domain of $f+g$ is the set of all x that lie in the domain of f and the domain of g. The domain of f is the set of points x such that $5 - x \geqslant 0$, or in other words $(-\infty, 5]$. And the domain of g is the set of points x such that $x^2 - 4 \geqslant 0$, or in other words $(-\infty, -2] \cup [2, \infty)$. The intersection of the two domains is then the intersection of $(-\infty, 5]$ with $(-\infty, -2] \cup [2, \infty)$, and this is exactly $(-\infty, -2] \cup [2, 5]$.

(b) $(f-g)(x) = \sqrt{5-x} - \sqrt{x^2-4}$. The domain is $(-\infty, -2] \cup [2, 5]$.

(c) $(fg)(x) = \sqrt{5-x}\sqrt{x^2-4}$. The domain is $(-\infty, -2] \cup [2, 5]$.

(d) $(f/g)(x) = \frac{\sqrt{5-x}}{\sqrt{x^2-4}}$. The domain is $(-\infty, -2) \cup (2, 5]$.

12. In every part, the domain is $\mathbb{R}$.

(a) $(f \circ g)(x) = 3x^2 - 1$

(b) *Worked Solution:* $(g \circ f)(x) = g(f(x)) = g(3x+2) = (3x+2)^2 - 1 = 9x^2 + 12x + 3$. The domain consists of all x such that $f(x)$ is defined and $g(f(x))$ is defined. Since both g and f are defined for all x, it follows that $g \circ f$ is too, hence its domain is $\mathbb{R}$.

(c) $(g \circ g)(x) = x^4 - 2x^2$.

(d) $(f \circ f)(x) = 9x + 8$.

13. In every part, the domain is $\mathbb{R}$.

(a) $(f \circ g)(x) = \sin(2x+2)$.

(b) $(g \circ f)(x) = 2\sin x + 2$.

(c) $(g \circ g)(x) = g(g(x)) = g(2x+2) = 2(2x+2) + 2 = 4x + 6$. Its domain consists of all x for which $g(x)$ and $g(g(x))$ are defined. Since $g(x)$ is defined for all x, it follows that $g \circ g$ is as well. Hence its domain is $\mathbb{R}$.

(d) $(f \circ f)(x) = \sin(\sin(x))$.

14. **(a)** $(f \circ g)(x) = x + 2$, and its domain is $\{x \mid x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$.

(b) *Worked Solution:* $(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)+2} = \frac{1}{\frac{1}{x}+2} = \frac{x}{2x+1}$. Its domain consists of all x for which $f(x)$ and $g(f(x))$ are defined. In other words, it consists of all x for which $x \neq 0$ and $1/x \neq -2$. And that means that it consists of all x for which $x \neq 0$ and $x \neq -1/2$. So the domain is $\{x \mid x \neq 0, x \neq -1/2\} = (-\infty, -1/2) \cup (-1/2, 0) \cup (0, \infty)$.

(c) $(g \circ g)(x) = \frac{x+2}{2x+5}$. Its domain is $\{x \mid x \neq -2, x \neq -5/2\}$.

(d) $(f \circ f)(x) = x$, and its domain is $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

15. **(a)** $f(8)$ does not exist because f is not defined at 8.

(b) $\lim_{x\to 4^-} f(x)$ does not exist because we may find $x < a$ arbitrarily close to a such that $f(x) = 6$, *and* we may find $x < a$ arbitrarily close to a such that $f(x) = 2$.

(c) $\lim_{x\to 4^+} f(x) = 4$.

(d) $\lim_{x\to 4} f(x)$ does not exist for the same reasons that $\lim_{x\to 4^-} f(x)$ does not exist.

(e) $\lim_{x\to 0^+} f(x) = 2$.

(f) $\lim_{x\to 0^-} f(x) = 6$.

(g) $\lim_{x\to 0} f(x)$ does not exist because $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

(h) $f(0) = 2$.

(i) $\lim_{x\to -4^-} f(x) = 2$.

(j) $\lim_{x\to -4^+} f(x) = 4$.

(k) $f(-4) = 3$.

(l) $\lim_{x\to 8} f(x) = 6$.

16. **(a)** $\lim_{x\to -4^-} f(x) = \infty$.

(b) $\lim_{x\to -4^+} f(x) = -\infty$.

(c) $\lim_{x\to -4} f(x)$ does not exist.

(d) $\lim_{x\to 4^-} f(x) = -\infty$.

(e) $\lim_{x\to 4^+} f(x) = -\infty$.

(f) $\lim_{x\to 4} f(x) = -\infty$.

17. **(a)** *Worked Solution:* We use the fact that $\lim_{x\to -1} h(x)$ exists if and only if $\lim_{x\to -1^-} h(x)$ and $\lim_{x\to -1^+} h(x)$ both exist and are equal, in which case

$$\lim_{x\to -1} h(x) = \lim_{x\to -1^-} h(x) = \lim_{x\to -1^+} h(x).$$

Since $h(x) = 2 + x$ for $x < -1$ we have

$$\lim_{x\to -1^-} h(x) = \lim_{x\to -1^-} 2 + x = 2 + (-1) = 1$$

and since $h(x) = x^2$ for $-1 < x < 1$ we have

$$\lim_{x\to -1^+} h(x) = \lim_{x\to -1^+} x^2 = (-1)^2 = 1.$$

Since these limits both exist and are equal, we have

$$\lim_{x \to -1} h(x) = 1.$$

(b) The limit does not exist.

(c) *Worked Solution:*

$\lim_{x \to a} f(x)$ exists for all a except possibly $x = -1$ and $x = 1$.

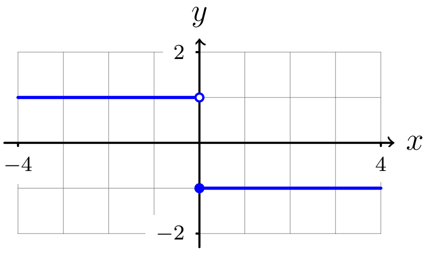
Now $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-}(-2 - x) = -1$ and $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+}(x^3) = -1$. Since $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$, it follows that $\lim_{x \to -1} f(x)$ exists.

Next, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 = 1$, while $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 1 - x = 0$. Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, it follows that $\lim_{x \to 1} f(x)$ does not exist.

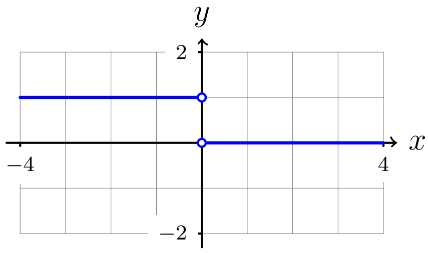
So $\lim_{x \to a} f(x)$ exists for all a except $a = 1$.

(d) The limit $\lim_{x \to a} f(x)$ exists for all x except $a = 0$.

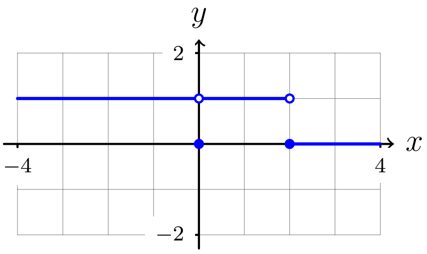
18. **(a)**



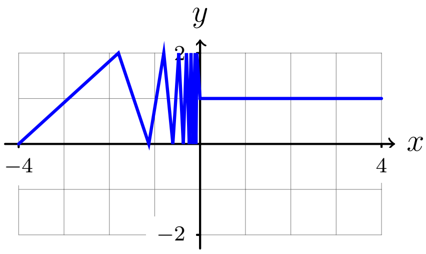
(b)



(c)



(d)



19. **(a)** $\displaystyle\lim_{x \to \pi/2^-} \tan x = \infty$ **(b)** $\displaystyle\lim_{x \to \pi/2^+} \tan x = -\infty$ **(c)** $\displaystyle\lim_{x \to 3^+} \frac{1+x}{x-3} = \infty$

(d) $\displaystyle\lim_{x \to 3^-} \frac{1+x}{x-3} = -\infty$ **(e)** $\displaystyle\lim_{x \to 0} \frac{1+x}{x^2(x-3)} = -\infty$

(f) *Worked Solution:*

As x approaches 3 the terms $1 + x$ and x^2 approach 4 and 9 respectively, while $x - 3$ approaches 0. Thus $\frac{1+x}{x^2(x-3)}$ grows without bound as x approaches 3.

However, if $x > 3$ then the term $1 + x$ is positive, as are x^2 and $x - 3$. Thus $\frac{1+x}{x^2(x-3)}$ is always positive for $x > 3$.

It follows that $\displaystyle\lim_{x \to 3^+} \frac{1+x}{x^2(x-3)} = \infty.$

20. **(a)** 6

(b) 16

(c) $-\sqrt[3]{3}$

(d) *Worked Solution:* First, $\lim_{x\to 1} 4h(x) = 4\lim_{x\to 1} h(x) = 4\times -3 = -12$, by the law for functions multiplied by a scalar. Since this limit exists and is nonzero, and since $\lim_{x\to 1} f(x)$ also exists, we may use the limit law for quotients to see that $\lim_{x\to 1} \frac{f(x)}{4h(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} 4h(x)} = \frac{2}{-12} = -1/6$.

(e) 0

21. **(a)** 31

(b) 15

(c) *Worked Solution:* For $u = 3$ we have $u^3 - 3u + 3 = 21$. So for u close to 3 we have $u^3 - 3u + 3 > 0$, and so we may use the n-th root law to see that

$$\lim_{u\to 3} \sqrt{u^3 - 3u + 3} = \sqrt{\lim_{u\to 3}(u^3 - 3u + 3)}$$

and by the direct substitution law this is equal to $\sqrt{3^3 - 3u + 3} = \sqrt{21}$.

(d) $\sqrt{\frac{3}{5}}$

22. The left hand side of the first equation is only defined for $x \neq 3$, while the right hand side is defined for all x. This is nevertheless enough to show that the two limits are equal.

23. **(a)** 1

(b) Does not exist.

(c) 7/4

(d) Does not exist.

(e) -4

(f) 12

(g) Does not exist.

(h) $1/\sqrt{2}$

(i) *Worked Solution:* First we simplify the function for $t \neq 0$:

$$\frac{1}{t} - \frac{2}{t^2 + 2t} = \frac{t+2}{t^2+2t} - \frac{2}{t^2+2t} = \frac{t+2-2}{t^2+2t} = \frac{1}{t+2}$$

Thus

$$\lim_{t\to 0}\left(\frac{1}{t} - \frac{2}{t^2+2t}\right) = \lim_{t\to 0}\left(\frac{1}{t+2}\right) = \frac{1}{2}.$$

(j) 1/54

24. *Worked Solution:*

(a) We will show that the left and right handed limits both exist and are zero. Indeed, since $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x \leqslant 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

and

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

so that $\lim_{x\to 0} |x|$ exists and is equal to 0.

(b) We do this in three parts, depending on the value of x.

First suppose that $x = 0$. Then $-|x| = 0$, $f(x) = 0$ and $|x| = 0$. So then $-|x| \leqslant f(x) \leqslant |x|$ certainly holds.

Next suppose that $x > 0$. In this case $|x| = x$. Then since $-1 \leqslant \cos(1/x) \leqslant 1$ and $x > 0$, we have $-x \leqslant x\cos(1/x) \leqslant x$. In other words, $-|x| \leqslant f(x) \leqslant |x|$.

Finally suppose that $x < 0$. In this case $|x| = -x$. Since $-1 \leqslant \cos(1/x) \leqslant 1$ and since $x < 0$, multiplying the inequality by x reverses the inequalities, so we have $-x \geqslant x\cos(1/x) \geqslant x$, or in other words $x \leqslant x\cos(1/x) \leqslant -x$, or in other words $-|x| \leqslant f(x) \leqslant |x|$, as required.

(c) We know that $\lim_{x\to 0}(-|x|) = 0$, $\lim_{x\to 0} |x| = 0$, and $-|x| \leqslant f(x) \leqslant |x|$ for all x. So the squeeze theorem applies and shows us that $\lim_{x\to 0} f(x) = 0$ as required.

25. **(a)** *Worked Solution:* Let us define functions p and q by $p(x) = x^2 - 5x + 8$ and $q(x) = 2x^2 - 11x + 17$. Then the question tells us that $p(x) \leqslant f(x) \leqslant q(x)$. Since p and q are polynomials, direct substitution shows that $\lim_{x\to 3} p(x) = p(3) = 2$ and $\lim_{x\to 3} q(x) = q(3) = 2$. So the squeeze theorem tells us that $\lim_{x\to 3} f(x) = 2$.

(b) 1

26. **(a)** Does not exist.

(b) 3

(c) *Worked Solution:* For $x < 0$ we have $\frac{|x|-3}{x+3} = \frac{-x-3}{x+3} = -1$ so that $\lim_{x\to -3}\left(\frac{|x|-3}{x+3}\right) = \lim_{x\to -3} -1 = -1$.

28. *Worked Solution:* We have $\lim_{x\to 0}\left(\dfrac{g(x)}{x^2}\right) = \lim_{x\to 0}\left(x \cdot \dfrac{g(x)}{x^3}\right) = \lim_{x\to 0}(x) \cdot \lim_{x\to 0}\left(\dfrac{g(x)}{x^3}\right) = 0 \cdot 4 = 0$. Similarly, $\lim_{x\to 0} \frac{g(x)}{x} = 0$.

29. $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$ serve as answers to both parts.

30. The example asks you to show that $\lim_{x\to a} f(x) = L$ where $a = 2$, $f(x) = 2x + 3$ and $L = 7$. So given $\epsilon > 0$ we must specify a $\delta > 0$ and show that if $|x - 2| < \delta$ then $|(2x + 3) - 7| < \epsilon$.

So the error is that the solution proves $|(2x + 3) - 7| < 2\epsilon$. To make a correct solution we can choose a different value of δ and follow through the same reasoning. In this case $\delta = \epsilon/2$ works. (If you can't see why this works, then try defining $\delta = c \cdot \epsilon$ and working through the steps of the solution, then check that by choosing $c = 1/2$ the computation ends as "$< \epsilon$".

31. The question asks us to show that $\lim_{x\to a} f(x) = L$, where $a = 3$, $f(x) = 3 - 5x$, and $L = -12$. So given $\epsilon > 0$, we must define a $\delta > 0$ and show that when $|x - a| < \delta$ we have $|f(x) - L| < \epsilon$.

The solution seems to do this. However there are two wrong steps: first, $|(-5)(3-x)| = |-5| \times |3 - x| = 5|3 - x|$, so there is a sign error in the third =, and $|3 - x| = |x - 3|$, so there is another sign error in the fourth =. Nothing else needs to be changed.

32. **(c)** *Pre-Solution:* Given $\epsilon > 0$ we must find $\delta > 0$ such that:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \left|(\frac{1}{2}x - 3) - (-2)\right| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \left|\frac{1}{2}x - 1\right| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \frac{1}{2}\,|x - 2| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } |x - 2| < 2\epsilon$$

If we choose $\delta = 2\epsilon$ then this will certainly be true.

Solution: Given $\epsilon > 0$, let $\delta = 2\epsilon$. Then if $0 < |x - 2| < \delta$, we have

$$\left|\left(\frac{1}{2}x - 3\right) - (-2)\right| = \left|\frac{1}{2}x - 1\right| = \frac{1}{2}\,|x - 2| < \frac{1}{2}\delta = \frac{1}{2}2\epsilon = \epsilon.$$

Thus $\lim\limits_{x\to 2} \left(\frac{1}{2}x - 3\right) = -2$ as claimed.

33. **(a)** *Solution:*

Given $\epsilon > 0$, let $\delta = \epsilon$. Then if $0 < |x - a| < \delta$, we have $|x - a| < \delta = \epsilon$. So $\lim\limits_{x\to a} x = a$.

(f) *Pre-Solution:* Given $\epsilon > 0$, we must find $\delta > 0$ such that:

$$\text{if } 0 < |x - (-5)| < \delta, \text{ then } |x^2 - 25| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x + 5| < \delta, \text{ then } |(x - 5)(x + 5)| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x+5| < \delta, \text{ then } |x-5| \cdot |x+5| < \epsilon$$

So we have to choose δ so that if $|x+5| < \delta$ then $|x-5|$ *and* $|x+5|$ are small enough. Let's suppose that $\delta \leqslant 1$. Then if $0 < |x+5| < \delta$, we have $0 < |x+5| < 1$, so that $-1 < x+5 < 1$, and consequently $-11 < x-5 < -9$, and consequently $|x-5| < 11$.

So we know that if $\delta \leqslant 1$ and $0 < |x-(-5)| < \delta$, then $|x^2-25| = |x-5| \cdot |x+5| < 11\delta$. Then if $\delta \leqslant \epsilon/11$, that will be enough. We can arrange this by taking $\delta = \min(1, \epsilon/11)$.

Solution: Given $\epsilon > 0$, let $\delta = \min(1, \epsilon/11)$. Suppose that $0 < |x-(-5)| < \delta$, or in other words that $0 < |x+5| < \delta$. Then $|x+5| < 1$, so that $-1 < x+5 < 1$, and consequently $-11 < x-5 < -9$, and consequently $|x-5| < 11$. So

$$|x^2-25| = |(x-5)(x+5)| = |x-5| \cdot |x+5| < 11 \cdot \delta \leqslant 11 \cdot \epsilon/11 = \epsilon.$$

Or in other words

$$|x^2-25| < \epsilon.$$

We have shown that $\lim\limits_{x \to -5} x^2 = 25$ as required.

34. **(a)** If $a \leqslant b$ then $\min(a,b) = a$. Since $a \leqslant a$ and $a \leqslant b$, we have $\min(a,b) \leqslant a$ and $\min(a,b) \leqslant b$ as required. If $b < a$ then a similar argument shows that the same inequalities hold.

The inequalities are used to show that $\delta \leqslant 2$ in the second paragraph, and to show that $\delta \leqslant 3\epsilon$ in the long series of inequalities.

(b) The identity is used to show that $|2-x| = |-(x-2)| = |x-2|$.

(c) If $|x-2| < 1$ then $-1 < x-2 < 1$, and adding 3 to all terms gives $2 < x+1 < 4$, so that $|x+1| > 2$ and consequently $\frac{1}{|x+1|} < \frac{1}{2}$.

If $|x-2| < 3$ then we find that $|x+1| > 0$, but then we can conclude nothing about $\frac{1}{|x+1|}$.

(d) Given $\epsilon > 0$, define $\delta = \min(1, 6\epsilon)$. Suppose that $0 < |x-2| < \delta$.

Since $|x-2| < \delta$ and $\delta \leqslant 1$, we have $|x-2| < 1$. It follows that $-1 < x-2 < 1$. By adding 3 to all terms we find that $2 < x+1 < 4$. Consequently $|x+1| > 2$, and rearranging gives $\frac{1}{|x+1|} < \frac{1}{2}$.

Now

$$\begin{aligned}\left|\frac{1}{x+1}-\frac{1}{3}\right| &= \left|\frac{3-(x+1)}{3(x+1)}\right| \\ &= \left|\frac{2-x}{3(x+1)}\right| \\ &= \frac{1}{3}\cdot\frac{1}{|x+1|}\cdot|2-x| \\ &= \frac{1}{3}\cdot\frac{1}{|x+1|}\cdot|x-2| \\ &< \frac{1}{3}\cdot\frac{1}{2}\cdot\delta \\ &\leqslant \frac{1}{3}\cdot\frac{1}{2}\cdot 6\epsilon \\ &= \epsilon\end{aligned}$$

and so $\left|\dfrac{1}{x+1}-\dfrac{1}{3}\right| < \epsilon$. Thus $\lim_{x\to 2}(\frac{1}{x+1}) = \frac{1}{3}$ as required.

(e) In this case you cannot prove an inequality of the form $\frac{1}{|x+1|} <?$ and so the solution cannot be made to work in this case.

35. **(a)** *Solution:*

Let $\epsilon > 0$. Define $\delta = \min(1, \epsilon/4)$ and let x be such that $0 < |x-2| < \delta$.

Since $\delta = \min(1, \epsilon/4)$ it follows that $\delta \leqslant 1$ so that $|x-2| < 1$. Thus $-1 < x-2 < 1$, so that $2 < x+1 < 4$, so that $|x+1| < 4$.

Now

$$\begin{aligned}|(x^2-x-3)-(-1)| &= |x^2-x-2| \\ &= |(x+1)(x-2)| \\ &= |x+1|\cdot|x-2| \\ &< 4\cdot\delta \\ &\leqslant 4\cdot\epsilon/4 \\ &= \epsilon\end{aligned}$$

so that overall we have $|(x^2-x-3)-(-1)| < \epsilon$ as required.

(d) *Solution:*

Let $\epsilon > 0$. Define $\delta = \min(1, \frac{12\epsilon}{5})$ and let x be such that $0 < |x-1| < \delta$.

Since $\delta = \min(1, \frac{12\epsilon}{5})$ it follows that $\delta \leqslant 1$, so that $|x-1| < 1$, so that $-1 < x-1 < 1$, so that $3 < x+3 < 5$, so that $|x+3| > 3$ and consequently $\frac{1}{|x+3|} < \frac{1}{3}$.