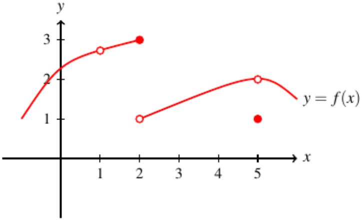
Now we will study a certain class of 'nice' functions that are called 'continuous'. These are the functions for which limits can be worked out by direct substitution. Intuitively, they are the functions whose graph can be drawn without taking the pen from the page.

Definition 1.76 (Continuity of a function at a point). *A function f is called continuous at a number a if $\lim_{x \to a} f(x) = f(a)$. In practice, for f to be continuous requires the following three things.*

1. *$f(a)$ is defined, or in other words, a is in the domain of f.*
2. *$\lim_{x \to a} f(x)$ exists and is finite.*
3. *$\lim_{x \to a} f(x) = f(a)$.*

If f is not continuous at a, then we say that f is discontinuous at a, or that it has a discontinuity at a.

Example 1.77. *Let f be the function with the following graph.*



At which of the following numbers a is f discontinuous? In each case, say which of properties 1, 2, and 3 fails.

- $a = 1$
- $a = 2$
- $a = 5$

Solution

- At $a = 1$ the function is discontinuous because $f(1)$ is not defined, so that property 1 fails.
- At $a = 2$ the function is discontinuous because, although $f(2)$ is defined, $\lim_{x\to 2} f(x)$ does not exist, so that property 2 fails.
- At $a = 5$ the function is discontinuous because, although $f(5)$ is defined, and although $\lim_{x\to 5} f(x)$ exists, the two are not equal. Indeed, $f(5) = 1$ but $\lim_{x\to 5} f(x) = 2$. So property 3 fails.

Example 1.78. *At which numbers are the functions f, g and h defined as follows discontinuous?*

- $f(x) = \dfrac{x^2 - x - 2}{x - 2}$
- $g(x) = \begin{cases} 1/x^2 & \textit{if } x \neq 0 \\ 1 & \textit{if } x = 0 \end{cases}$
- $h(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \textit{if } x \neq 2 \\ 1 & \textit{if } x = 2 \end{cases}$
- $H(x) = \begin{cases} 0 & \textit{if } x < 0 \\ 1 & \textit{if } x \geqslant 0 \end{cases}$

Solution

- f is not defined at 2, and so 2 is a discontinuity. But f is continuous everywhere else.
- g is discontinuous at 0 because, although $g(0)$ is defined, $\lim_{x\to 0} g(x)$ does not exist as a finite limit. But g is continuous everywhere else.
- h is continuous at every number except possibly 2. At $a = 2$, we see that $h(2)$ is defined and equal to 1, and that

$$\begin{aligned}\lim_{x\to 2} h(x) &= \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} \\ &= \lim_{x\to 2}(x+1) \\ &= 3\end{aligned}$$

  so that $\lim_{x\to 2} h(x)$ does exist, but is not equal to $h(2)$, so that $x = 2$ is in fact a discontinuity.
- For $a < 0$ we have $H(a) = 0 = \lim_{x\to a} H(x)$ and H is continuous there. For $a > 0$ we have $H(a) = 1 = \lim_{x\to a} H(x)$ and H is also continuous there. We have seen that the one sided limits of $H(x)$ at $x = 0$ do not agree, and therefore $\lim_{x\to 0} H(x)$ does not exist, and $H(x)$ is therefore not continuous there.

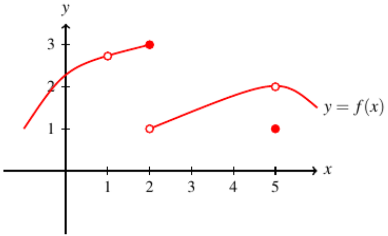
Definition 1.79 (Continuity of a function, at a point, from one side). *We say that a function f is continuous from the left at a number a if $\lim_{x\to a^-} f(x) = f(a)$. In other words, for f to be continuous from the left at a we require the following three properties:*

1. *$f(a)$ is defined.*
2. *$\lim\limits_{x\to a^-} f(x)$ exists and is finite.*
3. *$\lim\limits_{x\to a^-} f(x) = f(a)$.*

And we say that a function f is continuous from the right at a number a if $\lim_{x\to a^+} f(x) = f(a)$. In other words, for f to be continuous from the right at a we require the following three properties:

1. *$f(a)$ is defined.*
2. *$\lim\limits_{x\to a^+} f(x)$ exists and is finite.*
3. *$\lim\limits_{x\to a^+} f(x) = f(a)$.*

Example 1.80. *Let f again be the function with the following graph.*



Then

- f *is not continuous from the right or the left at* $a = 1$ *since* $f(1)$ *is not defined.*
- f *is continuous from the left at* $a = 2$ *because* $\lim_{x\to 2^-} f(x) = 3 = f(2)$.
- f *is not continuous from the right at* $a = 2$ *because* $\lim_{x\to 2^+} f(x) = 1 \neq 3 = f(2)$.
- f *is not continuous from the right or the left at* $a = 5$ *because* $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 2$ *but* $f(5) = 1$.

Definition 1.81 (Continuity of a function on an interval). *A function f is continuous on an interval I if it is continuous at a for all $a \in I$. If a is an endpoint of the interval, then we only require continuity from the left (if a is the right-hand end of the interval) or from the right (if a is the left-hand end of the interval).*

Example 1.82. *Show that the function f defined by $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.*

Solution If a lies in the range $-1 < a < 1$, then by the limit laws

$$\begin{aligned}\lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) &= 1 - \lim_{x \to a} \sqrt{1 - x^2} \\ &= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \\ &= 1 - \sqrt{1 - a^2} \\ &= f(a)\end{aligned}$$

so that f is continuous at a as required. Here, the use of the square root law required the fact that $1 - x^2 > 0$ for x close to, but not equal to, a. Similar computations with left and right limits show continuity from the right and left at -1 and 1 respectively. So f is continuous on $[-1, 1]$.

Now we will see how to construct new continuous functions from existing ones.

Theorem 1.83. *Suppose that f and g are continuous at a, and that c is a constant. Then the following functions are also continuous at a:*

$f+g \qquad f-g \qquad fg \qquad cf \qquad f/g$, *as long as* $g(a) \neq 0$.

Similarly, if f and g are continuous on an interval I, then the following are also continuous on I.

$f+g \qquad f-g \qquad fg \qquad cf \qquad f/g$, *as long as* $g(x) \neq 0$ *for all* x

Proof for $f+g$.

Let us show that if f and g are continuous at a, then so is $f+g$. Indeed, we have the following,

$$\begin{aligned}
\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\
&= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\
&= f(a)+g(a) \\
&= (f+g)(a)
\end{aligned}$$

so that $f+g$ is continuous at a. □

Using this theorem, we are able to construct a large supply of continuous functions.

Theorem 1.84. *Any polynomial function is continuous on* $\mathbb{R} = (-\infty, \infty)$*. Any rational function is continuous at any point in its domain, and so in particular is continuous on any interval in its domain.*

Proof.

The function f_0 defined by $f_0(x) = 1$ is continuous on $\mathbb{R}$, as is the function f_1 defined by $f_1(x) = x$. Then by the last theorem the function $f_1 f_1$, which is defined by $(f_1 f_1)(x) = x^2$ is also continuous on $\mathbb{R}$. Similarly, it follows that all of the functions f_n defined by $f_n(x) = x^n$, are continuous on $\mathbb{R}$. But then an arbitrary polynomial p with formula

$$p(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0$$

is nothing other than

$$p = c_k f_k + \cdots + c_1 f_1 + c_0 f_0,$$

which is continuous on $(-\infty, \infty)$, again using the previous theorem. Finally, an arbitrary rational function r defined by $r(x) = p(x)/q(x)$ where p and q are polynomials is nothing other than the quotient function p/q, which again by the previous theorem is continuous at all a for which $q(a) \neq 0$, or in other words, at any a that lies in its domain. □

Example 1.85. *On which intervals is the function* p *defined by* $p(x) = \frac{x^3+2x^2-1}{5-3x}$ *continuous?*

Solution p is a rational function, and so is continuous on any interval inside its domain. The domain of p contains every number for which $5 - 3x \neq 0$, i.e. for which $x \neq 5/3$, hence is $(-\infty, 5/3) \cup (5/3, \infty)$. So the largest possible intervals on which p is continuous are $(-\infty, 5/3)$ and $(5/3, \infty)$, and it is also continuous on any interval inside these.

In fact, we can do better than the previous theorem about continuity of polynomials and rational functions.

Theorem 1.86. *The following classes of functions are continuous at every point of their domains, and on every interval within their domains.*

- *Polynomials*
- *Rational functions*
- *Root functions*
- *Trigonometric functions (namely sin, cos, tan, sec, cosec, cot)*
- *Exponential functions*
- *Logarithmic functions*

(We will learn more about the last three classes of functions later on in the course.)

Example 1.87. *On which intervals is the function continuous?*

1. $f(x) = x^{100} - 2x^{37} + 75$
2. $g(x) = \dfrac{x^2 + 2x + 17}{x^2 - 1}$
3. $h(x) = \sqrt{x} + \dfrac{x+1}{x-1} - \dfrac{x+1}{x^2+1}$

Solution

1. f is polynomial, so is continuous on $(-\infty, \infty)$.
2. g is rational, so is continuous at every point on its domain, which is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$, so is continuous on the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.
3. h is a sum of:
   - A root function with domain $[0, \infty)$.
   - A rational function with domain $(-\infty, 1) \cup (1, \infty)$.
   - A rational function with domain $(-\infty, \infty)$.

   So h is continuous at any point a that lies in all three domains, and is continuous on the intervals in all three domains, i.e. $[0, 1)$ and $(1, \infty)$.

Here we have specified the largest possible intervals on which the functions are continuous. It follows that they are also continuous on any smaller intervals. For example, g is continuous on $(-10, 1)$ or $(-10, 1.5]$.

Now we will see one last way of constructing new continuous functions from old ones.

Theorem 1.88.

1. *Suppose that* $\lim_{x\to a} g(x) = b$ *and that* $\lim_{x\to b} f(x) = c$. *Then* $\lim_{x\to a}(f\circ g)(x)$ *exists and is equal to* c.
2. *Suppose that* $\lim\limits_{x\to a} g(x) = b$ *and that* f *is continuous at* b. *Then*
$$\lim_{x\to a}(f\circ g)(x) = f\left(\lim_{x\to a} g(x)\right).$$
3. *If* g *is continuous at* a *and* f *is continuous at* $g(a)$, *then* $f\circ g$ *is continuous at* a.

Example 1.89. *Let F be the function defined by $F(x) = \dfrac{1}{\sqrt{x} - 4}$. At what points is F continuous?*

Solution $F = p \circ q$, where p and q are defined by $q(x) = \sqrt{x}$ and $p(x) = \frac{1}{x-4}$. These are continuous on their domains, and so the same is true of F. Now

$$\mathrm{dom}(F) = \{x \mid x \geq 0 \text{ and } \sqrt{x} - 4 \neq 0\},$$

and

$$\sqrt{x} - 4 \neq 0 \iff \sqrt{x} \neq 4 \iff x \neq 16$$

So F is continuous at all non-negative numbers a except for $a = 16$.

Example 1.90. *Let f be the function defined as follows.*

$$f(x) = \begin{cases} x & \text{if } x \leqslant 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x-3} & \text{if } 3 \leqslant x \end{cases}$$

At which numbers is f discontinuous?

Solution First of all we show that f is continuous away from the 'joins' in the definition of f. More precisely, we show that f is continuous on each of the intervals $(-\infty, 1)$, $(1, 3)$ and $(3, \infty)$.

- On the interval $(-\infty, 1)$, f is given by $f(x) = x$. This is a polynomial, so f is continuous at every point of $(-\infty, 1)$.
- On the interval $(1, 3)$, f is given by $f(x) = 1/x$. This is a rational function and $(1, 3)$ lies in its domain, so f is continuous at every point of $(1, 3)$.
- On the interval $(3, \infty)$, f is given by $f(x) = \sqrt{x-3}$, so $f = p \circ q$ where p and q are defined by $q(x) = x - 3$ and $q(x) = \sqrt{x}$. Now if $x \in (3, \infty)$ then q is continuous at x, and p is continuous at $q(x) > 0$, so that f is continuous at x.

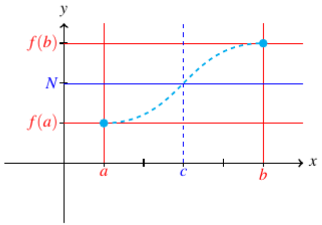
It remains to see whether f is continuous at $a = 1$ and $a = 3$. In these cases we understand $\lim_{x \to a} f(x)$ by looking at the one-sided limits, which we can compute using the relevant 'piece' of the definition of f.

- $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x = 1$, and $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (1/x) = 1$, so $\lim_{x \to 1} f(x) = 1$. But $f(1) = 1$ as well, so that f is continuous at 1.
- $\lim_{x \to 3-} f(x) = \lim_{x \to 3-} (1/x) = \frac{1}{3}$, and $\lim_{x \to 3+} f(x) = \lim_{x \to 3+} \sqrt{x-3} = 0$, so f is not continuous at 3.

Now we will see an important result about continuous functions. It is the result that says that the graph of a continuous function does not contain any vertical 'gaps' or 'jumps'.

Theorem 1.91 (The Intermediate Value Theorem). *Suppose that f is continuous on the closed interval $[a, b]$, that $f(a) \neq f(b)$, and that N is a number lying between $f(a)$ and $f(b)$, but not equal to either $f(a)$ or $f(b)$. Then there is $c \in (a, b)$ such that $f(c) = N$.*

To try to explain this theorem, we depict the information on the following graph



Here the only thing we know about the function f is that it is continuous, and that it passes through $(a, f(a))$ and $(b, f(b))$. What the theorem guarantees is that the graph actually crosses the line $y = N$ at (c, N), as opposed to somehow magically jumping over the line $y = N$. The theorem is not entirely straightforward, and we will not give the proof.

Example 1.92. *Show that the equation* $4x^3 - 6x^2 + 3x - 2 = 0$ *has a solution between* 1 *and* 2.

Solution Remember that a *solution* of an equation in a variable x is a choice of x that makes that equation true. So here, a solution is a choice of x for which $4x^3 - 6x^2 + 3x - 2$ really equals 0.

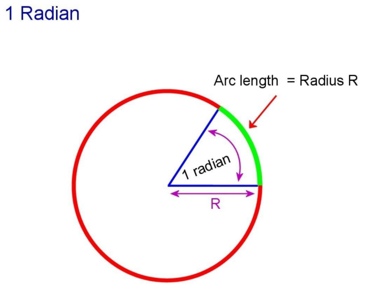
We will apply the intermediate value theorem with the function f defined by $f(x) = 4x^3 - 6x^2 + 3x - 2$, with $a = 1$ and $b = 2$, and with $N = 0$. Let's check that the conditions are satisfied. First, f is a polynomial, and so is continuous on $(-\infty, \infty)$, and in particular is continuous on $[a, b] = [1, 2]$. Next, $f(a) = f(1) = 4 - 6 + 3 - 2 = -1$ and $f(b) = f(2) = 32 - 24 + 6 - 2 = 12$, so that $N = 0$ does lie between $f(a)$ and $f(b)$, but is not equal to either of them. So the intermediate value theorem applies and shows that there is $c \in (a, b) = (1, 2)$ for which $f(c) = N$, i.e. $f(c) = 0$. This c is the required solution of the original equation.

Example 1.93. *Show that the equation* $\sin^2(x) = \frac{1}{5}$ *has a solution between* 0 *and* $\frac{\pi}{4}$*.*

Solution Again, a solution to this equation is a choice of x that makes the equation true. Define f by $f(x) = \sin^2(x)$. Now $\sin$ is continuous on its domain $(-\infty, \infty)$, and $f(x) = \sin^2(x) = (\sin(x))^2$, so that f is also continuous on $(-\infty, \infty)$, and in particular f is continuous on $[0, \pi/4]$. Now $f(0) = 0$ and $f(\pi/4) = 1/2$. Let $N = 1/5$, then N lies between $f(0)$ and $f(\pi/4)$, and is not equal to either of them. So the intermediate value theorem applies and shows that there is $c \in (0, \pi/4)$ such that $f(c) = 1/5$, or in other words that $\sin^2(c) = 1/5$. Thus c is the required solution to the original equation.

Continuity of the trigonometric functions

We will give now a concrete proof of the fact that the trigonometric functions $\sin(x)$ and $\cos(x)$ are continuous. Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ it will also be continuous in its domain of definition. A similar statement holds for $\cot(x)$. One important thing to mention here: when we write $\sin(x)$, the angle for the sine function is measured in *radians*, and not in *degrees*. An angle of x radians will have $\frac{180x}{\pi}$ degrees, and an angle of r degrees will have $\frac{r\pi}{180}$ radians. The radians have the following interpretation: In a circle of radius 1, an angle of x radians corresponds to an arc of length x. See the picture below. Use the picture also to see why $\sin(x) < x$ when $0 < x < \pi/2$. This inequality depends on measuring in radians and not in degrees.



We prove the continuity of $\sin(x)$ and $\cos(x)$ in a few steps:

1. $\sin(x)$ is continuous at $x = 0$. For $0 < x < \pi/2$ it holds that $0 < \sin(x) < x$. By the squeeze Theorem it follows that $\lim_{x\to 0^+} \sin(x) = 0 = \sin(0)$. In a similar way we can also show that $\lim_{x\to 0^-} \sin(x) = 0$, or simply by using the fact that $\sin(x)$ is an odd function.
2. $\cos(x)$ is continuous at $x = 0$. This is true because for $-\pi/2 < x < \pi/2$ it holds that $\cos(x) = \sqrt{1 - \sin(x)^2}$, and we already know that $\sin(x)$ is continuous at 0.
3. $\sin(x)$ is continuous everywhere. Let a be any number. It holds that $\lim_{x\to a} \sin(x) = \lim_{x\to 0} \sin(a + x)$ (explain why!). But $\sin(a + x) = \sin(a)\cos(x) + \sin(x)\cos(a)$. By using now the continuity of $\sin(x)$ and $\cos(x)$ at zero, and using the fact that $\sin(0) = 0$ and $\cos(0) = 1$, we get that this limit is $\sin(a)$. This implies that $\sin$ is continuous on the entire $\mathbb{R}$.