1-5 The precise definition of the limit

Definition 1.63 (The precise definition of limit). *Let f be a function defined on some open interval containing a, except possibly at a itself. Then we say that the limit of f as x approaches a is L, and write*

$$\lim_{x \to a} f(x) = L,$$

if the following condition holds.

> *For every number $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.*

Question The following questions are supposed to help you to understand the definition of the limit.

1. The set of all y satisfying $|y - L| < \epsilon$ is an interval. Which interval?
2. Think of ϵ as a small number. Describe in words what it means for y to satisfy the condition $|y - L| < \epsilon$. Do not use ϵ in your answer.
3. Now think of δ as a small number. Describe in words what it means for x to satisfy $0 < |x - a| < \delta$. Do not use δ in your answer.

Solution

1. It is the interval $(L - \epsilon, L + \epsilon)$.
2. It means that y is close to L.
3. It means that x is close to a but not equal to a.

So now let's try to understand the definition of $\lim_{x \to a} f(x) = L$. The original version is:

For every number $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

And we think of it as saying:

We can make $f(x)$ as close to L as we like by making x close enough to a, but not necessarily equal to a.

Let us understand what exactly we need to show when we want to prove that

$$\lim_{x \to a} f(x) = L.$$

The definition of the limit has the following form:
For every number $\epsilon > 0$ there is a number $\delta > 0$ such that SOMETHING HAPPENS.
At first glance, it looks as if there are infinitely many statement that we have to prove here, one statement for each possible value of ϵ: We have to prove that
For $\epsilon = 0.5$ there is a number $\delta > 0$ such that Something happens
and that
For $\epsilon = 0.1$ there is a number $\delta > 0$ such that Something happens
and that
For $\epsilon = 0.02$ there is a number $\delta > 0$ such that Something happens
and so on. However, we are going to prove all these statements at once. Here is an example:

Example 1.64. *Use the precise definition of the limit to show that* $\lim_{x \to 5} x = 5$.

Solution We write first exactly what we need to prove:
For every number $\epsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.
Here $f(x) = x$, $a = 5$ and $L = 5$. So we need to prove the following statement:
For every number $\epsilon > 0$ there is a number $\delta > 0$ such that $|x - 5| < \epsilon$ whenever $0 < |x - 5| < \delta$.
We need to prove this statement for every $\epsilon > 0$. So for $\epsilon = 0.1$ we need to find a number $\delta > 0$ such that if $0 < |x - 5| < \delta$ then $|x - 5| < 0.1 = \epsilon$. Such a number is $\delta = \epsilon = 0.1$. For $\epsilon = 0.01$ we need to fine a number $\delta > 0$ such that if $0 < |x - 5| < \delta$ then $|x - 5| < 0.01 = \epsilon$. Such a number is $\delta = \epsilon = 0.01$. We see that we do not have to go through all possible values of ϵ. We just need to say what δ we choose, given ϵ. In this case, we see that we can always choose $\delta = \epsilon$. Notice that usually the value of δ depends on ϵ.
So to conclude this, let us see what we need to do in order to show that $\lim_{x \to 5} x = 5$: We start with a number $\epsilon > 0$. We choose $\delta = \epsilon$. Then we have that $|x - 5| < \epsilon$ whenever $0 < |x - 5| < \delta$. This finishes the proof.

Example 1.65. *Use the precise definition of the limit to show that* $\lim_{x\to 0} 1 = 1$.

Solution In this case we have $a = 0$, $L = 1$ and $f(x) = 1$ the constant function. Notice that $f(x) - L = 1 - 1 = 0$ is constant zero, and does not depend on x. So to prove the limit is 1, we need to prove the following:
For every number $\epsilon > 0$ there is a number $\delta > 0$ such that $0 < \epsilon$ whenever $|x| < \delta$.
But it is always true that $0 < \epsilon$, and therefore we can choose whatever value of $\delta > 0$ we want. The statement will be true.

Let us see a bit more complicated example:

Example 1.66. *Use the precise definition of the limit to show that* $\lim_{x\to 6}(3x-4)=14$.

Solution We give here a general outline of how to write the solution to the question "Use the precise definition of the limit to show that $\lim\limits_{x\to a} f(x)=L$."

Let $\epsilon>0$. *Choose* $\delta=\cdots$. *Suppose that* $0<|x-a|<\delta$. *Then*

$$\begin{aligned}|f(x)-L| &= \cdots\\ &= \cdots\\ &= \cdots\\ &< \cdots\\ &= \cdots\\ &= \epsilon.\end{aligned}$$

Thus $|f(x)-L|<\epsilon$ *as required.*

This is what a general answer should look like, at least for a simple question like this one. (Probably it doesn't feel simple right now, but you'll get used to it!) Now let's start to make the answer our own by putting in the specifics. We are answering the question "Use the precise definition of the limit to show that $\lim_{x \to 6}(3x - 4) = 14$" and so we have $f(x) = 3x - 4$, $a = 6$ and $L = 14$. So now let's write out the next version of the answer.

Let $\epsilon > 0$. Choose $\delta = \cdots$. Suppose that $0 < |x - 6| < \delta$. Then

$$\begin{aligned} |(3x - 4) - 14| &= \cdots \\ &= \cdots \\ &= \cdots \\ &< \cdots \\ &= \cdots \\ &= \epsilon. \end{aligned}$$

Thus $|(3x - 4) - 14| < \epsilon$ as required.

Now what we have to do is fill in all of the blanks, which are currently denoted by $\cdots$. We start, perhaps surprisingly, by filling in the blanks after $|(3x - 4) - 14| = \cdots$.

Here is how it goes:

Let $\epsilon > 0$*. Choose* $\delta = \cdots$*. Suppose that* $0 < |x - 6| < \delta$*. Then*

$$\begin{aligned} |(3x - 4) - 14| &= |3x - 18| \\ &= |3(x - 6)| \\ &= 3|x - 6| \\ &< 3\delta \\ &= \cdots \\ &= \epsilon. \end{aligned}$$

Thus $|(3x - 4) - 14| < \epsilon$ *as required.*

What did we just do? We simplified the expression $|(3x - 4) - 14|$ until it contained the expression $|x - 6|$, and then we used the assumption that $|x - 6| < \delta$. Now we have to fill in the rest. Ask yourself: what value of δ can we choose so that we can fill in the rest? We want to fill in the equations $3\delta = \cdots = \epsilon$. What value of δ gives us $3\delta = \epsilon$? It is $\delta = \epsilon/3$. So now we can complete the solution:

Let $\epsilon > 0$*. Choose* $\delta = \epsilon/3$*. Suppose that* $0 < |x - 6| < \delta$*. Then*

$$\begin{aligned} |(3x-4) - 14| &= |3x - 18| \\ &= |3(x-6)| \\ &= 3|x-6| \\ &< 3\delta \\ &= 3(\epsilon/3) \\ &= \epsilon. \end{aligned}$$

Thus $|(3x-4) - 14| < \epsilon$ *as required.*

This is the final answer!

Remark 1.67. *Here are some very important points for you to remember.*

- *If an exam or test question wants you to use the precise definition of the limit, it will say so. Otherwise you should stick to the methods we have used so far.*
- *When you are asked a question like this in an exam, you are not expected to write out multiple versions like we did in the last solution. That was just my attempt to show you the steps. Instead, you will learn to write out your solution in one go by simply leaving things blank until you know what to write there.*
- *You will quickly learn that the hardest part of solving a question like this is to find or choose δ. However, your task when answering this question is to write out the whole solution.*
- *How you present your answer, and the order you present it, is important. Exam and test questions have marks for that. The simplest and best approach is to present your answer exactly as we have done here.*

We would like to see some more examples of calculation of the limit using the precise definition. For this, we first go through properties of the absolute value functions and inequalities.

Properties of the absolute value function. Here are some useful properties of the absolute value function that should come in useful through the course, especially in this section.

- $|a \cdot b| = |a| \cdot |b|$.
- $|a \cdot b| = a \cdot |b|$ if $a \geqslant 0$.
- $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ assuming that $b \neq 0$.
- $|a + b| \leqslant |a| + |b|$.
- $|x| < M$ if and only if $-M < x < M$.
- More generally, $|x - a| < M$ if and only if $a - M < x < a + M$.

Properties of inequalities.

And here are some useful properties of inequalities.

- If $a < b < c$ then $a < c$.
- If $a \leqslant b < c$ then $a < c$.
- If $a < b \leqslant c$ then $a < c$.
- If $a \leqslant b \leqslant c$ then $a \leqslant c$.
- If $a < b$ and $c > 0$ then $ac < bc$.
- If $a < b$ and $c < 0$ then $bc < ac$. In particular $-b < -a$.
- If $a < b$ and the sign of a and b is the same (that is: they are both positive or both negative) , then $\frac{1}{b} < \frac{1}{a}$.
- If a and b are positive numbers, $a < b$, and n is a natural number, then $a^n < b^n$ and $\sqrt[n]{a} < \sqrt[n]{b}$.
- If a and b are negative numbers, $a < b$, and n is a natural number, then $a^n < b^n$ whenever n is odd, and $b^n < a^n$ whenever a is even (Why is there a difference between even and odd values of n?).

Example 1.68. *Use the precise definition of the limit to show that* $\lim_{x\to 3}(4x+5) = 17$.

Solution Let's follow the same process as before. I am going to write out what my solution looks like at each step, so that you can see what I do. But when you write out your answer, you should simply leave the blanks as gaps to fill in, and you should only write out the solution once! I start with my skeleton answer.

Let $\epsilon > 0$. *Define* $\delta =$ *blank 1*. *Suppose that* $0 < |x-3| < \delta$. *Thus*

$$\begin{aligned} |(4x+5)-17| &= \textit{blank 3} \\ &\vdots \\ &= \epsilon. \end{aligned}$$

Thus $|(4x+5)-17| < \epsilon$ *as required.*

Now we will fill in as much of blank 3 as we can.

Let $\epsilon > 0$. *Define* $\delta =$ *blank 1*. *Suppose that* $0 < |x - 3| < \delta$. *Thus*

$$\begin{aligned} |(4x+5) - 17| &= |4x - 12| \\ &= |4(x-3)| \\ &= 4|x-3| \\ &= \textit{blank 3} \\ &\vdots \\ &= \epsilon. \end{aligned}$$

Thus $|(4x+5) - 17| < \epsilon$ *as required.*

I see now that in blank 3 I have an expression involving only $|x - 3|$. So I substitute in the inequality $|x - 3| < \delta$.

Let $\epsilon > 0$. *Define* $\delta =$ *blank 1. Suppose that* $0 < |x-3| < \delta$. *Thus*

$$\begin{aligned} |(4x+5)-17| &= |4x-12| \\ &= |4(x-3)| \\ &= 4|x-3| \\ &< 4\delta \\ &= \textit{blank 3} \\ &\vdots \\ &= \epsilon. \end{aligned}$$

Thus $|(4x+5)-17| < \epsilon$ *as required.*

Now I am in a position to choose my value of δ so that indeed $4\delta = \epsilon$, which is what we would need to complete blank 3. The correct choice is $\delta = \epsilon/4$. So now I can complete the solution:

Let $\epsilon > 0$. *Define* $\delta = \epsilon/4$. *Suppose that* $0 < |x-3| < \delta$. *Thus*

$$\begin{aligned}
|(4x+5)-17| &= |4x-12| \\
&= |4(x-3)| \\
&= 4|x-3| \\
&< 4\delta \\
&= 4(\epsilon/4) \\
&= \epsilon.
\end{aligned}$$

Thus $|(4x+5)-17| < \epsilon$ *as required.*

And that's it! **Please remember**, you should only write out the final answer, and you should do it by leaving the blanks as actual spaces on the page.

Example 1.69. *Use the precise definition of the limit to show that* $\lim_{x\to 0} x^2 = 0$.

Solution This example is a lot like the last one, so I won't spell out all of the steps. Here's the answer.

Let $\epsilon > 0$. *Define* $\delta = \sqrt{\epsilon}$. *Suppose that* $0 < |x| < \delta$. *Then*

$$\begin{aligned} |x^2 - 0| &= |x|^2 \\ &< \delta^2 \\ &= (\sqrt{\epsilon})^2 \\ &= \epsilon \end{aligned}$$

so that $|x^2 - 0| < \epsilon$ *as required.*

Now we're going to move on to some harder examples. These will need a slightly more elaborate solution. So let's imagine that we've been given the following typical question.

> Use the precise definition of the limit to show that $\lim_{x \to a} f(x) = L$. (Here f, L and a will be specified.)

The typical solution to a question like this looks like the following.

> Let $\epsilon > 0$. Define $\delta =$ blank 1. Suppose that $0 < |x - a| < \delta$.
> Then blank 2.
> Thus
>
> $$\begin{aligned} |f(x) - L| &= \text{blank 3} \\ &\vdots \\ &= \epsilon. \end{aligned}$$
>
> So $|f(x) - L| < \epsilon$ as required.

Here blank 1 will be your chosen δ, usually a formula involving ϵ. And blank 2 will be a paragraph involving some preliminary computations. And finally, blank 3 will be a series of simplifications and substitutions like before.

How do you go about turning the typical question and typical answer into an actual solution? In particular, how do you fill in the blanks? Here is what you should do:

1. Write out the skeleton solution, filling in the values of $f(x)$, L and a, and leaving the blanks empty. Blank 2 may have to be an entire paragraph, and blank 3 may require a lot of lines.
2. Start on blank 3, where you work out and simplify $|f(x) - L|$, aiming for an expression involving $|x - a|$.
3. If the resulting expression involves x in some other way, you will need to fill in blank 2, which will consist of some preliminary working out, and then continue filling in blank 3. (We will see about this later on.)
4. Blank 3 will now express $|f(x) - L|$ in terms of $|x - a|$. Now substitute δ in place of $|x - a|$, making sure that you include the relevant inequality.
5. Look at blank 3 and make a good choice to fill in blank 1. It should be an expression of the form $\delta = \cdots$ where the right hand side is a formula involving ϵ.
6. Now complete blank 3, using your choice of δ.
7. You have finished!

Let's see several examples of this in action. Before we do, we need a new definition.

Definition 1.70 (The minimum). *Let p and q be real numbers. Then $\min(p, q)$ denotes p or q, whichever is the smaller. So for example $\min(\pi, 3) = 3$ since $3 < \pi$, and $\min(\pi, 4) = \pi$ since $\pi < 4$. Note that*

$$\min(p, q) \leqslant p$$

and

$$\min(p, q) \leqslant q.$$

(I won't give a proof of these inequalities. Think about the definition of $\min(p, q)$ and it will hopefully become clear.)

Example 1.71. *Use the precise definition of the limit to show that* $\lim_{x \to 2}(x^2 + x + 1) = 7$.

Solution I'll write out my solution several times, adding more detail every time. However in lectures you will see me write everything out just once, and that is what you should do! I start with my skeleton answer.

> *Let* $\epsilon > 0$. *Define* $\delta =$ *blank 1*. *Suppose that* $0 < |x - 2| < \delta$.
> *Then blank 2*.
> *So*
>
> $$|(x^2 + x + 1) - 7| = \textit{blank 3}$$
> $$\vdots$$
> $$= \epsilon.$$
>
> *Thus* $|(x^2 + x + 1) - 7| < \epsilon$ *as required.*

Now I will fill in as much of blank 3 as I can.

Let $\epsilon > 0$. Define $\delta =$ blank 1. Suppose that $0 < |x-2| < \delta$.
Then blank 2.
Thus

$$\begin{aligned} |(x^2+x+1)-7| &= |x^2+x-6| \\ &= |(x+3)(x-2)| \\ &= |x+3| \cdot |x-2| \\ &= \textit{blank 3} \\ &\;\vdots \\ &= \epsilon. \end{aligned}$$

Thus $|(x^2+x+1)-7| < \epsilon$ as required. Now here we see a problem: we've obtained an expression involving $|x-2|$ as we want, but there's an annoying factor $|x+3|$ that we can't control. Now we will complete blank 2 and also edit blank 1.

Let $\epsilon > 0$. *Define* $\delta = \min(1,$ *blank 1*$)$. *Suppose that* $0 < |x-2| < \delta$. *Then since* $\delta \leqslant 1$ *and* $|x-2| < \delta$, *we have* $|x-2| < 1$. *That means* $-1 < (x-2) < 1$, *so by adding* 5 *to every term we see that* $4 < (x+3) < 6$, *so that* $|x+3| < 6$.

Thus

$$\begin{aligned} |(x^2+x+1)-7| &= |x^2+x-6| \\ &= |(x+3)(x-2)| \\ &= |x+3| \cdot |x-2| \\ &< 6|x-2| \\ &= \textit{blank 3} \\ &\ \vdots \\ &= \epsilon. \end{aligned}$$

Thus $|(x^2+x+1)-7| < \epsilon$ *as required.*

Now we can continue much as before. Here is the final answer.

Let $\epsilon > 0$. *Define* $\delta = \min(1, \epsilon/6)$. *Suppose that* $0 < |x-2| < \delta$. *Then since* $\delta \leqslant 1$ *and* $|x-2| < \delta$, *we have* $|x-2| < 1$. *That means* $-1 < (x-2) < 1$, *so by by adding* 5 *to every term we see that* $4 < (x+3) < 6$, *so that* $|x+3| < 6$.
Thus

$$\begin{aligned} |(x^2+x+1)-7| &= |x^2+x-2| \\ &= |(x+3)(x-2)| \\ &= |x+3| \cdot |x-2| \\ &< 6|x-2| \\ &< 6\delta \\ &\leqslant 6(\epsilon/6) \\ &= \epsilon. \end{aligned}$$

Thus $|(x^2+x+1)-7| < \epsilon$ *as required.*

At this stage you should study our proof carefully and ask why each step follows from the next. Here are some pointers.

- since $\delta \leqslant 1$: This follows because by $\delta = \min(1, \epsilon/6)$, and we always have $\min(p, q) \leqslant p$ and $\min(p, q) \leqslant q$.
- we have $|x - 2| < 1$ so $-1 < (x - 2) < 1$: This is because saying $|a| < M$ is the same as saying that $-M < a < M$. (This was one of our useful facts about the absolute value function.)
- $4 < (x + 3) < 6$, so $|x + 2| < 6$: This follows because if $4 < (x + 2) < 6$, then in particular $-6 < (x + 2) < 6$, so that $|x + 2| < 6$. (Again using one of our useful facts about the absolute value function.)
- $6\delta \leqslant 6(\epsilon/6)$: Remember that $\delta = \min(1, \epsilon/6)$, so that $\delta \leqslant \epsilon/6$, so that $6\delta \leqslant 6(\epsilon/6)$.

Also, we made a lot of choices when we wrote out the proof. Why did we make these choices, and what choice did we have?

- Why did we choose $\delta = \min(1, \epsilon/6)$? Defining δ this way means that we get two facts, namely $\delta \leqslant 1$ and $\delta \leqslant \epsilon/6$. We used these two different facts in two different places — try to see where.
- Why did we choose the $\epsilon/6$ in $\delta = \min(1, \epsilon/6)$? This was chosen to make the final "tower" of working out correct, where we replace an expression involving δ with one involving ϵ. You choose $\epsilon/6$ to make sure that you end that calculation with an ϵ.
- Why did we choose the 1 in $\delta = \min(1, \epsilon/6)$? Well, the 1 told us that $\delta \leqslant 1$, which we used in the second paragraph to put a bound on $|x + 3|$. In fact, it didn't matter what number we chose here. It could have been any positive number. To see why, replace the 1 with a 9 and change the rest of the proof accordingly. What happens?
- Why did we add 5 to every term of $-1 < (x - 2) < 1$? In this paragraph we were trying to understand the quantity $|x + 3|$. What we already knew was that $-1 < (x - 2) < 1$, and to make this tell us something about $(x + 3)$ we added 5, which turns the $(x - 2)$ into the $(x + 3)$.

Example 1.72. *Use the precise definition of the limit to show that* $\displaystyle\lim_{x \to 2} \left(\frac{x+3}{x-1} \right) = 5.$

Solution Again, I will go through the example in many steps, writing each one out in full so that you can see the changes. In the lectures I will just do it in one go, leaving blanks until I know what goes where, and that is what you should do when working out your own answers. I will start by writing out the typical solution, filling in the values of a, $f(x)$ and L, and also working out as much of the final computation as I can. Again, my δ will be a minimum, so I'll put that in as well.

Let $\epsilon > 0$. *Define* $\delta = \min(-, -)$. *Suppose that* $0 < |x-2| < \delta$.
Then blank 2.
Thus

$$
\begin{aligned}
\left|\left(\frac{x+3}{x-1}\right) - 5\right| &= \left|\frac{(x+3)-5(x-1)}{x-1}\right| \\
&= \left|\frac{-4x+8}{x-1}\right| \\
&= \left|\frac{-4(x-2)}{x-1}\right| \\
&= 4 \cdot \left|\frac{x-2}{x-1}\right| \\
&= 4 \cdot \frac{|x-2|}{|x-1|} \\
&= 4 \cdot \frac{1}{|x-1|} \cdot |x-2| \\
&= \textit{blank 3} \\
&\ \ \vdots \\
&= \epsilon.
\end{aligned}
$$

So $\left|\left(\frac{x+3}{x-1}\right) - 5\right| < \epsilon$ *as required.*

The situation looks good, because we expressed $\left|\left(\frac{x+3}{x-1}\right)-5\right|$ in terms of $|x-2|$, but now there is a factor of $\frac{1}{|x-1|}$ making things complicated. Like in our previous example, we need to make sure that this factor of $\frac{1}{|x-1|}$ is not too large. That means that we need to make sure that $|x-1|$ itself is not too *small*. Here is how we do it.

> *Let* $\epsilon > 0$. *Define* $\delta = \min(1/2, -)$. *Suppose that* $0 < |x-2| < \delta$. *Then since* $\delta \leqslant 1/2$, *we have* $-1/2 < (x-2) < 1/2$. *Adding* 1 *to both sides gives us* $1/2 < (x-1) < 3/2$. *From this we can see that* $|x-1| > 1/2$. *This rearranges to tell us that* $\frac{1}{|x-1|} < 2$.
> *Thus*

$$\begin{aligned}
\left|\left(\frac{x+3}{x-1}\right)-5\right| &= \left|\frac{(x+3)-5(x-1)}{x-1}\right| \\
&= \left|\frac{-4x+8}{x-1}\right| \\
&= \left|\frac{-4(x-2)}{x-1}\right| \\
&= 4\cdot\left|\frac{x-2}{x-1}\right| \\
&= 4\cdot\frac{|x-2|}{|x-1|} \\
&= 4\cdot\frac{1}{|x-1|}\cdot|x-2| \\
&< 4\cdot 2\cdot|x-2| \\
&< 8\cdot\delta \\
&= \textit{blank 3} \\
&\ \vdots \\
&= \epsilon.
\end{aligned}$$

So $\left|\left(\frac{x+3}{x-1}\right)-5\right| < \epsilon$ *as required.*

Now all we have to do is fill in the second part of the minimum and complete our proof.

> *Let* $\epsilon > 0$. *Define* $\delta = \min(1/2, \epsilon/8)$. *Suppose that* $0 < |x - 2| < \delta$. *Then since* $\delta \leqslant 1/2$, *we have* $-1/2 < (x - 2) < 1/2$. *Adding* 1 *to both sides gives us* $1/2 < (x - 1) < 3/2$. *From this we can see that* $|x - 1| > 1/2$. *This rearranges to tell us that* $\frac{1}{|x-1|} < 2$.
> *Thus*

$$
\begin{aligned}
\left|\left(\frac{x+3}{x-1}\right)-5\right| &= \left|\frac{(x+3)-5(x-1)}{x-1}\right| \\
&= \left|\frac{-4x+8}{x-1}\right| \\
&= \left|\frac{-4(x-2)}{x-1}\right| \\
&= 4 \cdot \left|\frac{x-2}{x-1}\right| \\
&= 4 \cdot \frac{|x-2|}{|x-1|} \\
&= 4 \cdot \frac{1}{|x-1|} \cdot |x-2| \\
&< 4 \cdot 2 \cdot |x-2| \\
&= 8 \cdot |x-2| \\
&< 8 \cdot \delta \\
&\leqslant 8 \cdot (\epsilon/8) \\
&= \epsilon.
\end{aligned}
$$

So $\left|\left(\frac{x+3}{x-1}\right)-5\right| < \epsilon$ *as required.*

Like in the previous example, we made a lot of choices, and you might ask how and why we made those choices. The only point now that is different from before is this one:

- Why did we choose $1/2$ in $\delta = \min(1/2, \epsilon/8)$? The $1/2$ was used in the second paragraph in order to show that $1/2 < (x - 1) < 3/2$. The key thing here was to make sure that in the final inequality the two 'ends', namely $1/2$ and $3/2$, were both positive (or at least have the same sign). For this purpose, we needed $1/2$ to be small enough. Check this: Replace $1/2$ with $1/4$ and see what happens to the proof. Then replace it with 1 and see what happens.

Precise definition of infinite limits

We gave the precise definition of $\lim_{x \to a} f(x) = L$. We will also give now the precise definition of $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = L$. We will the give a few examples.

Definition 1.73. *Let f be a function defined close to a point a. We say that $\lim_{x \to a} f(x) = \infty$ if the following condition holds: For every natural number N there exists a $\delta > 0$ such that $f(x) > N$ whenever $|x - a| < \delta$.*

Let us compare again this definition with the imprecise definition we gave before. The imprecise definition was the following:
We say that $lim_{x \to a} f(x) = \infty$ if we can make $f(x)$ as large and positive as we wish by making x sufficiently close to, but not equal to, a.
Now let us write the above definition with colors:
We say that $\lim_{x \to a} f(x) = \infty$ if the following condition holds: For every natural number N there exists a $\delta > 0$ such that $f(x) > N$ whenever $0 < |x - a| < \delta$. The phrase "we can make $f(x)$ as large and positive as we wish" translates to saying that $f(x) > N$. The "by making x sufficiently close to.." part translates to $|x - a| < \delta$. Let us see an example:

Example 1.74. *Prove that* $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution In the proof here we will follow the pattern of the previous proofs. The only difference is that now we need to show $f(x) > N$ instead of $|f(x) - L| < \epsilon$. We use again the blanks as before. In first step we write: Let $N > 0$ be a natural number. Choose $\delta = \cdots$. Assume that $0 < |x| < \delta$. We have :

$$\begin{aligned} f(x) = \frac{1}{x^2} &= \cdots \\ &= \\ &\vdots \\ &> N. \end{aligned}$$

In our case, we know that $|x| < \delta$. By the inequality rules we have we know that:
$x^2 = |x|^2 < \delta^2$ and therefore
$f(x) = \frac{1}{x^2} > \frac{1}{\delta^2}$. We want to get $f(x) > N$. This will be true if $N = \frac{1}{\delta^2}$, or, in other words, if $\delta = \frac{1}{\sqrt{N}}$. So the proof of $\lim_{x\to 0} f(x) = \infty$ is this:
Let $N > 0$ be a natural number. Choose $\delta = \frac{1}{\sqrt{N}}$. Assume that $0 < |x| < \delta$. Then we have:

$$f(x) = \frac{1}{x^2} > \frac{1}{\delta^2} = N$$

and we are done.

In a similar way, we can also give precise definition for all the other limits, like the one sided limits, limits such as $\lim_{x \to \infty} f(x)$ and so on. The importance of this part in the course was for you to see that there is a precise definition for the intuitive notion of a limit.

The example above is the last example of computing a limit with the precise definition that we will do in class. Please study the examples in detail, and follow the hints, to try to understand how they work and how you yourself could answer them. Now we will see how the precise definition of the limit can be used to prove the limit laws, in this case the sum law. All of the limit laws we have seen so far can be proved in this way, though some proofs are harder than others.

Theorem 1.75 (The sum law for limits). *Let f and g be functions defined close to a, but not necessarily at a itself. Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then $\lim_{x\to a}[f(x)+g(x)]$ exists and*

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Proof.

Suppose that $\lim_{x\to a} f(x) = L$ and that $\lim_{x\to a} g(x) = M$. Then we must show that $\lim_{x\to a}[f(x)+g(x)] = L+M$.

Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there is $\delta_1 > 0$ such that $|f(x)-L| < \epsilon/2$ whenever $0 < |x-a| < \delta_1$. And since $\lim_{x\to a} g(x) = M$, there is $\delta_2 > 0$ such that $|g(x)-M| < \epsilon/2$ whenever $0 < |x-a| < \delta_2$. Define $\delta = \min(\delta_1, \delta_2)$ and suppose that $0 < |x-a| < \delta$. Then since $\delta \leqslant \delta_1$, we have $0 < |x-a| < \delta_1$, and so $|f(x)-L| < \epsilon/2$. And since $\delta \leqslant \delta_2$, we have $0 < |x-a| < \delta_2$, and so $|g(x)-M| < \epsilon/2$. So finally

$$\begin{aligned}
|(f(x)+g(x))-(L+M)| &= |(f(x)-L)+(g(x)-M)| \\
&\leqslant |f(x)-L| + |g(x)-M| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

so that $|(f(x)+g(x))-(L+M)| < \epsilon$ as required. $\square$