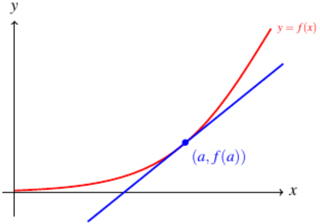
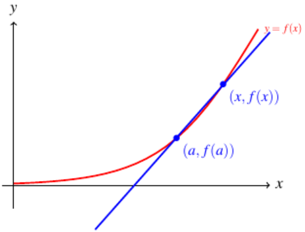
1-4 The limit of a function

Tangents

We are now going to move on to the subject of *limits*. In order to motivate this, we will talk about tangents. Suppose that we want to find the tangent to the graph $y = f(x)$ at a point $(a, f(a))$.



Here, the *tangent* is the line that passes through $(a, f(a))$ and has the same gradient as the curve at that point. The equation of the tangent line will then be $(y - f(a)) = m(x - a)$ where m is the gradient of $y = f(x)$ at $(a, f(a))$. How do we compute m? We can approximate m by choosing an x close to a and considering the *secant* line that passes through $(a, f(a))$ and $(x, f(x))$:



If x is close to a, then the secant line is a good approximation to the tangent line, and so the gradient m_x of the secant is a good approximation to m. Now

$$m_x = \frac{f(x) - f(a)}{x - a}.$$

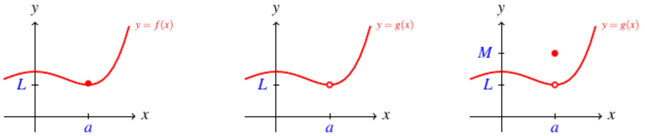
We would like to set $x = a$ and then obtain $m_a = m$. But the formula for m_x makes no sense in the case $x = a$. Nevertheless, it seems reasonable to expect that as x gets closer to a, m_x gets closer to m, and so we say that m is the *limit* of m_x as x approaches a. (This only works if f is nice enough.) In this part of the course we are going to study and understand this idea of limit.

Definition 1.27 (Limits — the imprecise definition). *Suppose that f is a function defined for all x near to, but not necessarily equal to, a number a. We say that the limit of $f(x)$ as x approaches a is L, and we write*

$$\lim_{x \to a} f(x) = L,$$

if we can make the values of $f(x)$ as close to L as we like by choosing x close enough to a. Here, the phrase "x near to a" means "x is in some open interval that contains a".

Example 1.28. *$f(a)$ does not have to be defined for $\lim_{x \to a} f(x)$ to exist, and even if $f(a)$ is defined, it may not be equal to $\lim_{x \to a} f(x)$. This is shown in the following three examples.*



$\mathrm{dom}(f) = (-\infty, \infty)$	$\mathrm{dom}(g) = (-\infty, 0) \cup (0, \infty)$	$\mathrm{dom}(h) = (-\infty, \infty)$
$f(a) = L$	a not in $\mathrm{dom}(g)$	$h(a) = M$
$\lim_{x \to a} f(x) = L$	$\lim_{x \to a} g(x) = L$	$\lim_{x \to a} h(x) = L$

Example 1.29. *Convince yourself that for every a it holds that*

$$\lim_{x\to a} 1 = 1 \textit{ and } \lim_{x\to a} x = a.$$

Example 1.30. *Investigate* $\lim_{x\to 0} \sin(\frac{\pi}{x})$.

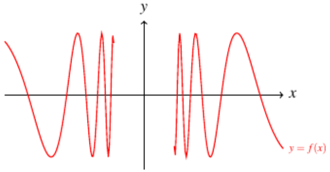
Solution Let f be the function defined by $f(x) = \sin(\frac{\pi}{x})$. Then we have:

$$f\left(\frac{2}{5}\right) = \sin\left(\frac{\pi}{2/5}\right) = \sin\left(\frac{5\pi}{2}\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\left(\pi/2\right) = 1$$
$$f\left(\frac{2}{9}\right) = \sin\left(\frac{\pi}{2/9}\right) = \sin\left(\frac{9\pi}{2}\right) = \sin\left(4\pi + \frac{\pi}{2}\right) = \sin\left(\pi/2\right) = 1$$
$$f\left(\frac{2}{13}\right) = \sin\left(\frac{\pi}{2/13}\right) = \sin\left(\frac{13\pi}{2}\right) = \sin\left(6\pi + \frac{\pi}{2}\right) = \sin\left(\pi/2\right) = 1$$

and so on. This shows that we can find numbers x that are as close to 0 as we wish with $f(x) = 1$. Similar calculations show that

$$f\left(\frac{2}{3}\right) = -1$$
$$f\left(\frac{2}{7}\right) = -1$$
$$f\left(\frac{2}{11}\right) = -1$$

and so on, so that we can find numbers x that are as close to 0 as we wish with $f(x) = -1$. This means that the limit *does not exist*: there is no one L for which $f(x)$ gets closer and closer to L as x gets closer and closer to 0. This is shown pretty clearly on the graph of $y = f(x)$. The graph oscillates ever more rapidly as you approach 0, so it does not get close any single value.

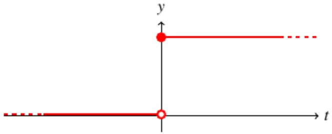


Example 1.31.

The Heaviside function H is defined as follows:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

Its graph is as follows.



In this case $\lim_{t \to 0} H(t)$ does not exist. This is because there are numbers t as close to 0 as we like with $H(t) = 0$ (any negative number) and numbers t as close to 0 as we like with $H(t) = 1$ (any positive number). So there is no single number L such that we can make $H(t)$ as close to L as we like by making t sufficiently close to 0.

Definition 1.32 (One-sided limits). *Let f be a function that is defined for all x close to, and less than, a number a. We write*

$$\lim_{x \to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left is L if we can make $f(x)$ as close to L as we wish by making x sufficiently close to, and less than, a. Here the phrase 'for all x close to and less than a' means 'for x in some open interval of the form (b, a)'.
Now let f be a function that is defined for all x close to, and greater than, a number a. We write

$$\lim_{x \to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right is L if we can make $f(x)$ as close to L as we wish by making x sufficiently close to, and greater than, a. Here the phrase 'for all x close to and greater than a' means 'for x in some open interval of the form (a, b)'.

The first limit is 'from the left' and its definition involves the phrase 'less than', while the second limit is 'from the right' and its definition involves the phrase 'greater than'. That is because if we draw x and a on the number line, then $x < a$ means x is to the left of a, while $x > a$ means that x is to the right of a.

Example 1.33. *Let H be the Heaviside function defined above by*

$$H(t) = \begin{cases} 0 & \textit{if } t < 0 \\ 1 & \textit{if } t \geqslant 0 \end{cases}$$

What are $\lim\limits_{t \to 0^-} H(t)$ and $\lim\limits_{t \to 0^+} H(t)$?

Solution $\lim_{t\to 0^-} H(t) = 0$. This is because, however close to 0 we would like $H(t)$ to be, we can achieve that by taking $t < 0$ close enough to 0. Indeed, for any $t < 0$ we have $H(t) = 0$.
Similarly, $\lim_{t\to 0^+} H(t) = 1$. This is because, however close to 1 we would like $H(t)$ to be, we can achieve that by taking $t > 0$ close enough to 0. Indeed, for any $t > 0$ we have $H(t) = 1$.

Here is an important rule about limits.

- The limit $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are equal, in which case

$$\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

This rule means that if $\lim_{x\to a^+} f(x)$ does not exist, or if $\lim_{x\to a^-} f(x)$ does not exist, or if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ do exist but are not equal, then $\lim_{x\to a} f(x)$ does not exist.

Example 1.34. *Let f be the function defined by $f(x) = \sin(\frac{\pi}{x})$. Then $\lim_{x\to 0^+} f(x)$ does not exist, for exactly the same reasons given before. So $\lim_{x\to 0} f(x)$ does not exist.*

Example 1.35. *$\lim_{t\to 0^-} H(t) = 0$ and $\lim_{t\to 0^+} H(t) = 1$ do both exist, but they are not equal. So $\lim_{t\to 0} H(t)$ does not exist.*

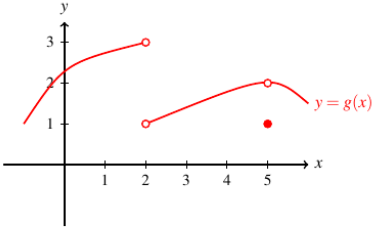
Example 1.36. *Show that* $\lim_{x \to 0} |x| = 0$.

Solution Remember that the absolute value function is defined by:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

The limit $\lim_{x \to 0^-} |x|$ depends only on the values of $|x|$ for $x < 0$. But if $x < 0$ then $|x| = -x$, and so $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-}(-x) = 0$. Similarly, the limit $\lim_{x \to 0^+} |x|$ depends only on the values of $|x|$ for $x > 0$. But if $x > 0$ then $|x| = x|$, and so $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$. Since $\lim_{x \to 0^+} |x|$ and $\lim_{x \to 0^-} |x|$ exist and are equal, we have $\lim_{x \to 0} |x| = 0$.

Example 1.37. *Let g be the function whose graph is as follows.*



State the values of the following, if they exist.

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$ (d) $g(2)$

(e) $\lim_{x\to 5^-} g(x)$ (f) $\lim_{x\to 5^+} g(x)$ (g) $\lim_{x\to 5} g(x)$ (h) $g(5)$

Solution

(a) 3 (b) 1 (c) does not exist. (d) $g(2)$ not defined

(e) 2 (f) 2 (g) 2 (h) 1

Example 1.38. *Define the function f as follows.*

$$f(x) = \begin{cases} x+1 & \text{if } x \leqslant 0 \\ 2x+1 & \text{if } 0 < x \leqslant 1 \\ x^2 & \text{if } 1 < x \end{cases}$$

Which limits exist, and what are their values?

(a) $\lim_{x \to 0} f(x)$ *(b)* $\lim_{x \to 1} f(x)$ *(c)* $\lim_{x \to \frac{1}{2}} f(x)$

Solution

(a) The piecewise definition means that we can always work out the left and right limits. For example, when we compute $\lim_{x\to 0^-} f(x)$, we can assume that x is close to 0 and less than 0, so that the first 'piece' of f applies and $f(x) = x + 1$. Thus

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1.$$

Similarly, when we compute $\lim_{x\to 0^+} f(x)$ we can assume that $x > 0$ and that x is close to 0, so that the second 'piece' of f applies and $f(x) = 2x + 1$. Thus

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2x+1) = 1.$$

Consequently, since the left and right limits exist and are both equal to 1, we have

$$\lim_{x\to 0} f(x) = 1.$$

(b) In this case we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 1) = 2 + 1 = 3$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1^2 = 1.$$

Since the left and right limits are different,

$$\lim_{x \to 1} f(x) \text{ does not exist.}$$

(c) In this case we have $\lim_{x \to 1/2} f(x)$. (Here $1/2$ is 'in the middle' of one of the pieces of f, rather than being a point where the definition changes.) In computing this limit we may assume that x is close to $1/2$, so that the middle 'piece' of f applies and

$$\lim_{x \to \frac{1}{2}} f(x) = \lim_{x \to \frac{1}{2}} (2x + 1) = 2 \times \frac{1}{2} + 1 = 2.$$

Calculating limits using the limit laws

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Let c be a constant. Then we have the following *limit laws*, which are rules for computing limits.

- **Sum rule.**
$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
- **Difference rule.**
$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$
- **Scalar rule.**
$$\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$$
- **Product rule.**
$$\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\cdot\left[\lim_{x\to a} g(x)\right]$$
- **Quotient rule.**
$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$
so long as $\lim_{x\to a} g(x) \neq 0$.

Remark 1.39. *These limit laws, and the ones that follow, all also work for one-sided limits. (In other words, take a limit law, and replace every instance of '$x \to a$' with '$x \to a^-$' or '$x \to a^+$' and the result is still a true law.*

Definition 1.40 (Polynomials and rational functions). *A polynomial is a function* p *defined by a formula of the form*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, \ldots, a_0$ *are real numbers. A rational function is a function* r *defined by a formula of the form*

$$r(x) = \frac{p(x)}{q(x)}$$

where p *and* q *are polynomials. The domain of any polynomial is* $\mathbb{R}$, *and the domain of the rational function* r *above is* $\{x \mid q(x) \neq 0\}$, *unless stated otherwise.*

Using the limit laws above, we see that if $p(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ is a polynomial, then

$$\lim_{x\to a} p(x) = b_n \lim_{x\to a} x^n + b_{n-1} \lim_{x\to a} x^{n-1} + \cdots + b_1 \lim_{x\to a} x + \lim_{x\to a} b_0 =$$

$$b_n(\lim_{x\to a} x)^n + b_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + b_1 \lim_{x\to a} x + \lim_{x\to a} b_0 =$$

$$b_n a^n + b_{n-1}a^{n-1} + \cdots b_1 a + b_0 = p(a).$$

Similarly, if q is another polynomial which satisfies $q(a) \neq 0$ then we have that

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} = \frac{p(a)}{q(a)}.$$

We summarize this:

- **Direct Substitution Law.** Let f be a rational function and let a lie in the domain of f. Then
$$\lim_{x\to a} f(x) = f(a).$$
(This rule applies to many more functions than just the rational functions, as we will see later in the course.)
- Let f and g be functions such that $f(x) = g(x)$ for x close to, but not necessarily equal to, a. Then
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$
Here, "close to a" means "in an open interval containing a".

Example 1.41. *Evaluate* $\lim_{x \to -2} \frac{x^2 - 4}{x^2 + 6x + 8}$.

Solution We can't use direct substitution to do this. If we tried it, we would find

$$\lim_{x \to -2} \left[\frac{x^2 - 4}{x^2 + 6x + 8} \right] = \frac{(-2)^2 - 4}{(-2)^2 + 6(-2) + 8} = \frac{0}{0}.$$

The result is nonsense, so we know that we went wrong somewhere. Indeed, -2 is not in the domain of the function, so that direct substitution was not permitted in the first place. However, our failed computation gives us a clue: substituting $x = -2$ makes $x^2 - 4$ and $x^2 + 6x + 8$ equal to 0, and consequently $(x - (-2)) = (x + 2)$ is a factor of both. In fact, for $x \neq -2$, we have

$$\frac{x^2 - 4}{x^2 + 6x + 8} = \frac{(x - 2)(x + 2)}{(x + 2)(x + 4)} = \frac{x - 2}{x + 4}.$$

So by the last limit law, we have

$$\lim_{x \to -2} \left[\frac{x^2 - 4}{x^2 + 6x + 8} \right] = \lim_{x \to -2} \left[\frac{(x - 2)(x + 2)}{(x + 2)(x + 4)} \right] = \lim_{x \to -2} \left[\frac{x - 2}{x + 4} \right] = \frac{-2 - 2}{-2 + 4} = -2.$$

In this computation, we were allowed to use direct substitution to compute $\lim_{x \to -2} \left[\frac{x-2}{x+4} \right]$ because -2 lies in the domain of the function. We know that -2 lies in the domain because when we substitute $x = -2$ the denominator is not 0.

Example 1.42. *Calculate the value of* $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.

Solution Observe that $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \neq 1$. Since the limit as x goes to 1 does not depend on what happens *at* 1, this is enough to show that $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1)$. Now $\lim_{x\to 1}(x+1) = 1 + 1 = 2$, by the limit laws.

Definition 1.43 (Infinite limits).
Let f be defined near to a, except possibly at a itself. We say

$$\lim_{x \to a} f(x) = \infty$$

if we can make $f(x)$ as large and positive as we wish by making x sufficiently close to, but not equal to, a. And we say

$$\lim_{x \to a} f(x) = -\infty$$

if we can make $f(x)$ as large and negative as we wish by making x sufficiently close to, but not equal to, a. We leave it to the reader to define the following similar notions.

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

Warning The symbol ∞ appears above only as part of the 'phrases' $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} f(x) = -\infty$. It does not appear elsewhere or in any other way. Remember, ∞ is not a number!

Example 1.44. $\lim_{x\to 0} \frac{1}{x^2} = \infty$. *This is because if* K *is a large positive number, then if we want to make sure that* $\frac{1}{x^2} > K$ *it is enough to make sure that* x *is in the range* $-\frac{1}{\sqrt{K}} < x < \frac{1}{\sqrt{K}}$ *and* $x \neq 0$.

Example 1.45. $\lim_{x\to 0^+} \frac{1}{x} = \infty$ *and* $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. *For if* x *is small and positive, then* $\frac{1}{x}$ *is large and positive, as large as we like if* x *is close enough to* 0, *so that* $\lim_{x\to 0^+} \frac{1}{x} = \infty$. *Similarly for the second case. Consequently,* $\lim_{x\to 0} \frac{1}{x}$ *does not exist, as a finite or infinite limit.*

Example 1.46. *Compute* $\displaystyle\lim_{x\to 3^+} \frac{1}{x-3}$ *and* $\displaystyle\lim_{x\to 3^-} \frac{1}{x-3}$.

Solution If x is close to 3, but greater than 3, then $x-3$ is small and positive, so $\frac{1}{x-3}$ is large and positive, and so $\lim_{x\to 3^+} \frac{1}{x-3} = \infty$. If x is close to 3, but less than 3, then $x-3$ is small and negative, so $\frac{1}{x-3}$ is large and negaive, and so $\lim_{x\to 3^-} \frac{1}{x-3} = -\infty$.

Example 1.47. *Let* f *be the function defined by* $f(x) = \frac{x^2-4x+3}{x^2-9}$. *Think about* $\lim_{x\to 3} f(x)$, $\lim_{x\to -3} f(x)$ *and, if necessary, the left and right versions of these limits.*

Solution First, let's consider what happens when we make a substitution.

- When we substitute $x = 3$, we find that $x^2 - 4x + 3 = 0$ and $x^2 - 9 = 0$.
- When we substitute $x = -3$, we find that $x^2 - 4x + 3 = 24$ and $x^2 - 9$=0.

Remember that if a polynomial becomes 0 when we substitute $x = a$, then you know that $(x - a)$ is a factor of that polynomial. Indeed,

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 9} = \frac{(x-3)(x-1)}{(x-3)(x+3)} = \frac{x-1}{x+3}$$

so long as x is not equal to 3 or -3. But the limits in question do not depend on what happens at $x = 3$ or $x = -3$, and so

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x-1}{x+3} = \frac{3-1}{3+3} = \frac{1}{3}$$

Now as x approaches -3 from the left, $x - 1$ will be approximately -4, and $x + 3$ will be small and negative. So $\frac{x-1}{x+3}$ will be large and positive. Thus

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \frac{x-1}{x+3} = \infty$$

And as x approaches -3 from the right, $x - 1$ will be approximately -4, and $x + 3$ will be small and positive. So $\frac{x-1}{x+3}$ will be large and negative. Thus

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \frac{x-1}{x+3} = -\infty.$$

And finally $\lim_{x \to -3} f(x)$ does not exist, either as a finite or an infinite limit.

Definition 1.48. *Let* f *be a function which is defined for all* x *which is positive and large enough. In other words: there exists an* $r \in \mathbb{R}$ *such that* $f(x)$ *is defined for all* $x > r$*. We say*

$$\lim_{x \to \infty} f(x) = L$$

if we can make $f(x)$ *close to* L *as we wish by taking* x *to be large enough.*
We define $\lim_{x \to -\infty} f(x) = L$ in a similar fashion. We next define infinte limits at infinity:

Definition 1.49. *Let* f *be a function which is defined for all* x *which is positive and large enough. We say*

$$\lim_{x \to \infty} f(x) = \infty$$

if we can make $f(x)$ *as large as we wish by taking* x *to be large enough.* We can similarly define

$$\lim_{x \to -\infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = -\infty, \text{ and } \lim_{x \to \infty} f(x) = -\infty.$$

Example 1.50. *Convince yourself that* $\lim_{x\to\infty} 1 = 1$ *and that* $\lim_{x\to\infty} x = \infty$.

Remark 1.51. *The limit laws still work if one replaces* a *with* ∞ *or* $-\infty$, *assuming that the limits exist and are finite. Thus, if* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to\infty} g(x)$ *exist and finite, then* $lim_{x\to\infty} f(x) + g(x)$ *exist and is equal to* $\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$.

Example 1.52. *Calculate the limit*

$$\lim_{x\to\infty} \frac{x+1}{2x+3}.$$

Solution Since $\lim_{x\to\infty}(x+1) = \lim_{x\to\infty} 2x+3 = \infty$ we can not just use the quotient rule. Indeed, we will receive the phrase $\frac{\infty}{\infty}$ which has no clear meaning. Instead, we will rewrite the function $f(x) = \frac{x+1}{2x+3}$ as

$$f(x) = \frac{\frac{1}{x}(1+\frac{1}{x})}{\frac{1}{x}(2+\frac{3}{x})}.$$

Notice that in order to write f in this way we need to assume that $x \neq 0$. Since we are studying the limit for $x \to \infty$ this does not matter for us, since we only care about the values of $f(x)$ for large values of x. The function f can thus be written as $f(x) = \frac{1+\frac{1}{x}}{2+\frac{3}{x}}$. So we wrote f as a quotient of two functions which do have finite limits at ∞. It holds that $\lim_{x\to\infty} 1 + \frac{1}{x} = 1$ and $\lim_{x\to\infty}(2+\frac{3}{x}) = 2$. It holds by the quotient rule that $\lim_{x\to\infty} f(x) = \frac{1}{2}$.

How to approach limit questions Let's imagine that we have the following imaginary question about a function f.

> **Question:** *Does* $\lim_{x \to a} f(x)$ *exist as a finite or infinite limit, and if so, what is its value?*

Here is how we try to solve this question:

- If f is given by a single formula, begin by substituting $x = a$.
- If $f(a)$ is defined, and f is sufficiently nice (see the Direct Substitution rule later; quotients of polynomials are always sufficiently nice) then $\lim_{x \to a} f(x) = f(a)$.
- If $f(a)$ is not defined, then attempt to simplify, and start the process again. (Note: If substituting $x = a$ into a polynomial produces 0, then $(x - a)$ is a factor of the polynomial.
- If f is defined piecewise and a lies at the 'join' of two pieces, then first consider $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

There is not always a systematic approach to computing limits, for example $\lim_{x \to 0} \frac{\sin(x)}{x}$, which we will compute later from first principles. Often you may have to try different approaches before finding the one that works. However, you will only be asked questions that you are capable of solving!

Example 1.53. *Suppose that f and g are functions defined near to, but not necessarily at, the number a, and suppose that $\lim_{x\to-2} f(x) = 1$ and $\lim_{x\to-2} g(x) = -1$. Use limit laws to evaluate the following.*

1. $\lim\limits_{x\to-2}[f(x) + 5g(x)]$
2. $\lim\limits_{x\to-2}\left[\dfrac{f(x)}{g(x)}\right]$

Solution

1.

$$\begin{aligned}
\lim_{x\to-2}[f(x) + 5g(x)] &= \lim_{x\to-2}[f(x) + [5g(x)]] \\
&= \lim_{x\to-2} f(x) + \lim_{x\to-2}[5g(x)] && \text{by sum law} \\
&= \lim_{x\to-2} f(x) + 5\cdot\lim_{x\to-2} g(x) && \text{by scalar law} \\
&= 1 + 5\cdot(-1) \\
&= -4.
\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x\to-2}\left[\frac{f(x)}{g(x)}\right] &= \frac{\lim_{x\to-2} f(x)}{\lim_{x\to-2} g(x)} && \text{by quotient law} \\
&= \frac{1}{-1} = -1
\end{aligned}$$

Remark 1.54. *In the last solution, we should really have checked that at each step, when we used a limit law, the limits in question existed. However, in each case this was checked by the remaining steps of the computation. So all is well, and we allow ourselves to tackle the problems in this way.*

Warning If you are computing a limit using limit laws, and arrive at an expression that makes no sense, or a limit that does not exist, then discard the computation and try something else.

Here are a further set of limit laws, some of which you have already seen before. We again assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and are finite, and that c is any number.

- **Power law.** Let n be a positive integer. Then:

$$\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$$

- **Constant.**

$$\lim_{x\to a} c = c.$$

- **Identity law.**

$$\lim_{x\to a} x = a.$$

- **Consequently**

$$\lim_{x\to a} x^n = a^n$$

  for any positive integer n.

- **And similarly**

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$$

  for any positive integer n, where $a > 0$ if n is even.

- **An more generally**

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

  where n is a positive integer and, if n is even, then $\lim_{x\to a} f(x) > 0$.

Remark 1.55. *The purpose of the previous computation was to show how the various different limit laws work. However, if we compare the first and fifth steps of the above computation we have the following.*

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \qquad \textit{and} \qquad \frac{[-2]^3 + 2[-2]^2 - 1}{5 - 3[-2]}$$

Looking at these two expressions, it seems that we should be able to go from one to the other by directly substituting $x = -2$. That is the message of the direct substitution law below.

Example 1.56. *Evaluate* $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$*.*

Solution Substituting $t = 0$ into the formula gives us $\frac{0}{0}$, so we need to do something else. Factorizing top and bottom of the fractions also would not help. Instead we use the following trick. **Try to understand and remember the trick!** For $t \neq 0$ we have:

$$\begin{aligned}
\frac{\sqrt{t^2+9}-3}{t^2} &= \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} \\
&= \frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2(\sqrt{t^2+9}+3)} \\
&= \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)} \\
&= \frac{t^2}{t^2(\sqrt{t^2+9}+3)} \\
&= \frac{1}{\sqrt{t^2+9}+3}
\end{aligned}$$

This means that

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}.$$

Here the last step follows from a version of direct substitution that we will see shortly.

Theorem 1.57. *Suppose that* $f(x) \leqslant g(x)$ *for all* x *close to, but not necessarily equal to,* a*. Then* $\lim_{x\to a} f(x) \leqslant \lim_{x\to a} g(x)$*, assuming that both limits exist.*

Warning You cannot replace the two instances of $\leqslant$ in the theorem with $<$, because then the theorem fails. Can you see an example of this?

Theorem 1.58 (Squeeze Theorem). *(aka the Sandwich Theorem) Suppose that*

$$f(x) \leqslant g(x) \leqslant h(x)$$

for x *close to, but not necessarily equal to,* a*. Suppose also that*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x).$$

Then $\lim_{x\to a} g(x)$ *exists and is equal to* L*.*

Example 1.59. *Show that* $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Solution Define $g(x) = x^2 \sin(1/x)$. (The squeeze theorem tells us about the function g, and we want to know about $x^2 \sin(1/x)$, which is why we made this choice.) Define $f(x) = -x^2$ and $h(x) = x^2$. Since $-1 \leqslant \sin(1/x) \leqslant 1$ and $x^2 \geqslant 0$, we have

$$-x^2 \leqslant x^2 \sin(1/x) \leqslant x^2.$$

In other words,

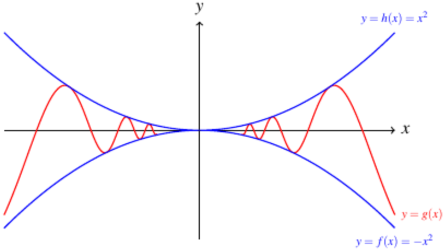
$$f(x) \leqslant g(x) \leqslant h(x).$$

Also, $\lim_{x \to 0} f(x) = \lim_{x \to 0}(-x^2) = 0$ and $\lim_{x \to 0} h(x) = \lim_{x \to 0}(x^2) = 0$. So the squeeze theorem applies (with f, g and h as specified above, with $a = 0$, and with $L = 0$), and tells us that $\lim_{x \to 0} x^2 \sin(1/x) = 0$.

Example 1.60. *Suppose you had been asked to find* $\lim_{x\to 0}|x|\cos(1/x)$ *using the squeeze theorem. You would put* $g(x) = |x|\cos(1/x)$*, because the squeeze theorem tells us about the limit of* g*. But what* f *and* h *would you choose?*

Solution We would choose $f(x) = -|x|$ and $h(x) = |x|$. Then the squeeze theorem could be applied, exactly as in the previous example, but this time using the inequalities $-1 \leqslant \cos(1/x) \leqslant 1$ and $|x| \geqslant 0$.

Example 1.61. *Here is a 'picture' of the squeeze theorem at work, in the case where* $g(x) = x^2 \sin(1/x)$ *as in Example 1.59. For this we took* $f(x) = -x^2$ *and* $h(x) = x^2$*. Here are the graphs of the three functions, with* $y = f(x)$ *at the bottom in blue and* $y = h(x)$ *at the top in blue, and* $y = g(x)$ *between them in red.*



The inequality $f(x) \leqslant g(x) \leqslant h(x)$ *translates to the fact that the red graph is squeezed between the two blue graphs. The conclusion of the squeeze theorem, that* $\lim_{x\to 0} g(x) = 0$*, is now immediately clear from the graphs.*

Example 1.62. *Show that* $\lim_{x\to 0} x\sin(1/x) = 0$.

Solution (Note: This example is harder than the previous two. The reason for this is that in the previous examples we had $x^2 \geqslant 0$ and $|x| \geqslant 0$ respectively, whereas now the analogous inequality $x \geqslant 0$ is not true.)
Define g by $g(x) = x\sin(1/x)$, and define f and h as follows.

$$f(x) = \begin{cases} -x & \text{if } x \geqslant 0 \\ x & \text{if } x < 0 \end{cases} \qquad \text{and} \qquad h(x) = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $-1 \leqslant \sin(1/x) \leqslant 1$, we have:

- If $x \geqslant 0$, then $-x \leqslant x\sin(1/x) \leqslant x$.
- If $x < 0$, then $-x \geqslant x\sin(1/x) \geqslant x$, or in other words, $x \leqslant x\sin(1/x) \leqslant -x$.

The first bullet point says that $f(x) \leqslant g(x) \leqslant h(x)$ when $x \geqslant 0$, and the second bullet point says that $f(x) \leqslant g(x) \leqslant h(x)$ when $x < 0$. So $f(x) \leqslant g(x) \leqslant h(x)$ holds for all x. But note that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} h(x) = 0$, as we see for example by inspecting the left and right hand limits of each one. So the squeeze theorem applies and tells us that $\lim_{x\to 0} g(x) = 0$ as required.

Limits: rules of thumb To conclude, we give here a summary of what is allowed, and what is not allowed, to do when calculating limits involving infinity. We write this here in an a slightly imprecise way. As we mentioned, we cannot treat ∞ as a number, so for each rule we say exactly what we mean:

- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then
$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x)g(x) = \infty.$$
We write this informally as $\infty + \infty = \infty$ and $\infty \cdot \infty = \infty$.
- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c > 0$, then
$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{g(x)} = \infty.$$
- If $\lim_{x\to a} f(x) = c$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.
- If $\lim_{x\to a} f(x) = c > 0$ and $\lim_{x\to a} g(x) = 0$, and $g(x) > 0$ for x close enough to a, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$. Similar results hold when $c < 0$, or $g(x) < 0$, or both. We just need to take care of the sign.
- Assume that $f(x) \leq g(x)$ when x is close enough to a number a. If $\lim_{x\to a} f(x) = \infty$ then $\lim_{x\to a} g(x) = \infty$.

Now for a list of problematic situations. In these situations we cannot calculate the limit directly using the limit rules, but we have to simplify the function somehow first.

- The limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ when $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x) = \infty$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. We say informally that $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are not defined.
- The limit $\lim_{x\to a} f(x) - g(x)$ where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. We say informally that $\infty - \infty$ is not defined.
- The limit $\lim_{x\to a} f(x)g(x)$ where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$. We say informally that $0 \cdot \infty$ is not defined.