1-3 New functions from old functions

Definition 1.19 (Sums, differences, products and quotients). *Let f and g be functions. The sum and difference of f and g, denoted by $f + g$ and $f - g$, are the new functions defined by*

$$(f + g)(x) = f(x) + g(x)$$

and

$$(f - g)(x) = f(x) - g(x).$$

The product and quotient of f and g, denoted by fg and f/g, are the new functions defined by

$$(fg)(x) = f(x)g(x)$$

and

$$(f/g)(x) = f(x)/g(x).$$

Example 1.20. *Let f and g be the functions defined by $f(x) = \sin(x)$ and $g(x) = 2^x$. Find formulas for $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $(f/g)(x)$.*

Solution

- $(f+g)(x) = f(x) + g(x) = \sin(x) + 2^x$
- $(f-g)(x) = f(x) - g(x) = \sin(x) - 2^x$
- $(fg)(x) = f(x)g(x) = \sin(x) \cdot 2^x$
- $(f/g)(x) = f(x)/g(x) = \frac{\sin(x)}{2^x}$

Definition 1.21 (Domains of sums, differences, products and quotients). *The domains of $f+g$, $f-g$, fg and f/g are defined, according to our domain convention, as the largest sets for which the given formulas make sense and produce a real number. Working this out gives us the following:*

$$\begin{aligned}\mathrm{dom}(f+g) &= \mathrm{dom}(f)\cap\mathrm{dom}(g)\\ \mathrm{dom}(f-g) &= \mathrm{dom}(f)\cap\mathrm{dom}(g)\\ \mathrm{dom}(fg) &= \mathrm{dom}(f)\cap\mathrm{dom}(g)\\ \mathrm{dom}(f/g) &= \{x\in\mathrm{dom}(f)\cap\mathrm{dom}(g)\mid g(x)\neq 0\}\end{aligned}$$

For example, in the case of $f+g$, the formula $f(x)+g(x)$ makes sense whenever $f(x)$ and $g(x)$ are both defined, since we can always add any two real numbers to get another real number, and so $\mathrm{dom}(f+g)=\mathrm{dom}(f)\cap\mathrm{dom}(g)$. On the other hand, in the case of f/g, the formula $f(x)/g(x)$ makes sense whenever f and g are both defined and $g(x)\neq 0$, since we cannot divide by 0.

Example 1.22. *Let f and g be the functions defined by $f(x) = 1/x$ and $g(x) = \sqrt{x} - 1$. What are the domains of the functions $f + g$, $f - g$, fg and f/g?*

Solution We do not need to work out the functions $f + g$, $f - g$, fg and f/g in order to answer this question! Instead, we will use the rules for their domains given in the last definition. In order to do this we first work out that $\mathrm{dom}(f) = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ and $\mathrm{dom}(g) = \{x \mid x \geqslant 0\} = [0, \infty)$. Thus we have:

- $\mathrm{dom}(f + g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = \{x \mid x \neq 0\} \cap [0, \infty) = (0, \infty)$.
- $\mathrm{dom}(f - g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = (0, \infty)$ as above.
- $\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = (0, \infty)$ as above.
- $\mathrm{dom}(f/g) = \{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \mid g(x) \neq 0\}$. Now $\mathrm{dom}(f) \cap \mathrm{dom}(g) = (0, \infty)$ as above, and $g(x) \neq 0$ means that $\sqrt{x} - 1 \neq 0$, i.e. that $x \neq 1$. So $\mathrm{dom}(f/g) = \{x \in (0, \infty) \mid x \neq 1\} = (0, 1) \cup (1, \infty)$.

Warning Never answer a question like "What is the domain of fg?" by first working out a formula for $(fg)(x)$ and then investigating when the formula makes sense and produces a real number. Do it like we did above, by first working out $\operatorname{dom}(f) \cap \operatorname{dom}(g)$ step-by-step.

Question f and g be the functions defined by

$$f(x) = x$$

and

$$g(x) = |x|.$$

1. What is the domain of f? Of g? Of f/g?
2. Write a 'piecewise' formula for f/g.

Solution

1. $\text{dom}(f) = \mathbb{R}$, $\text{dom}(g) = \mathbb{R}$, and

$$\begin{aligned} \text{dom}(f/g) &= \{x \in \text{dom}(f) \cap \text{dom}(g) \mid g(x) \neq 0\} \\ &= \{x \in R \cap R \mid |x| \neq 0\} \\ &= \{x \in \mathbb{R} \mid x \neq 0\} \\ &= (-\infty, 0) \cup (0, \infty). \end{aligned}$$

2. Remember that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

so then

$$(f/g)(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Let's check this. By the definition $(f/g)(x) = f(x)/g(x) = x/|x|$, so that if $x > 0$ then $|x| = x$ and $(f/g)(x) = x/x = 1$, and if $x < 0$ then $|x| = -x$ and $(f/g)(x) = x/(-x) = -1$.

Definition 1.23 (Composite functions). *If f and g are functions, the composite function $f \circ g$ is the new function defined by*

$$(f \circ g)(x) = f(g(x)).$$

In other words, to work out $(f \circ g)(x)$, we first apply g to work out $g(x)$, and then we apply f to the result to work out $f(g(x))$.

Example 1.24. *Let f and g be the functions defined by*

$$f(x) = x^2$$

and

$$g(x) = \sin(x).$$

Give formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution We just follow the definitions and substitute in when we can, so that

$$(f \circ g)(x) = f(g(x)) = f(\sin(x)) = (\sin(x))^2 = \sin^2(x)$$

and

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sin(x^2).$$

Note that $\sin^2(x)$ and $\sin(x^2)$ are not usually equal, so that $f \circ g$ and $g \circ f$ are not the same in this case. And in general, $f \circ g \neq g \circ f$. So you must take care when working out composites: make sure you do it in the right order!

Definition 1.25 (Domains of composite functions). *According to our domain convention, the domain of* $f \circ g$ *consists of all* x *for which the formula* $(f \circ g)(x) = f(g(x))$ *makes sense and produces a real number. Thus the domain consists of all* x *for which* $x \in \mathrm{dom}(g)$ *and* $g(x) \in \mathrm{dom}(f)$*. In other words:*

$$\mathrm{dom}(f \circ g) = \{x \mid x \in \mathrm{dom}(g) \textit{ and } g(x) \in \mathrm{dom}(f)\}.$$

Example 1.26. *Let f and g be defined by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$. Find the domains of $f \circ g$ and $g \circ g$.*

Solution To find $\mathrm{dom}(f \circ g)$ $\mathrm{dom}(g \circ g)$ we will use the rule given in the previous definition. To do this we need to know $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$, which are given by $\mathrm{dom}(f) = [0, \infty)$ and $\mathrm{dom}(g) = (-\infty, 2]$. So now

$$\begin{aligned}
\mathrm{dom}(f \circ g) &= \{x \mid x \in \mathrm{dom}(g) \text{ and } g(x) \in \mathrm{dom}(f)\} \\
&= \{x \mid x \leqslant 2 \text{ and } \sqrt{2-x} \geqslant 0\} \\
&= \{x \mid x \leqslant 2\} \\
&= (-\infty, 2].
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{dom}(g \circ g) &= \{x \mid x \in \mathrm{dom}(g) \text{ and } g(x) \in \mathrm{dom}(g)\} \\
&= \{x \mid x \leqslant 2 \text{ and } \sqrt{2-x} \leqslant 2\} \\
&= \{x \mid x \leqslant 2 \text{ and } 2 - x \leqslant 4\} \\
&= \{x \mid x \leqslant 2 \text{ and } x \geqslant -2\} \\
&= [-2, 2].
\end{aligned}$$

Warning Do not try to find $\mathrm{dom}(f \circ g)$ by first computing $(f \circ g)(x)$ and then working out the domain using the resulting formula. Why not? Because you could get the wrong answer. Let's see how.

Let a and b be the functions defined by $a(t) = b(t) = \frac{1}{t}$. Then $\mathrm{dom}(a) = \mathrm{dom}(b) = (-\infty, 0) \cup (0, \infty)$. So the domain of $a \circ b$ is

$$\begin{aligned} \mathrm{dom}(a \circ b) &= \{t \mid t \in \mathrm{dom}(b) \text{ and } b(t) \in \mathrm{dom}(a)\} \\ &= \{t \mid t \neq 0 \text{ and } \frac{1}{t} \neq 0\} \\ &= (-\infty, 0) \cup (0, \infty). \end{aligned}$$

On the other hand, $(a \circ b)(t) = a(1/t) = 1/(1/t) = t$, and this formula makes sense and produces a real number for all t. So if you used this formula to find the domain you would get the wrong answer.

So what is really happening? Well, when working out $(a \circ b)(t)$ we did some *simplification*, and this showed us that $f \circ g$ could be *extended* to include 0 in its domain.