1-2 Functions

Next, we introduce functions. Functions arise when one quantity depends on another, e.g.:

- The area A of a circle depends on the radius r of the circle. For each r there is just one value of A.
- The population P of the earth depends on the time t. At each time t, the population P is a number that can in principle be determined.

Definition 1.8 (Function). *A function f is a rule that assigns to each element x in a set D a unique element $f(x)$ in a set E. Usually, D and E will be sets of real numbers. The set D is called the domain of f, and the set E is called the codomain of f. We will sometimes write the domain of f as $\mathrm{dom}(f)$. The range of f is the set of all possible values of $f(x)$ as x goes through all elements of the domain (So the range is the 'smallest possible choice of E'). We write $\mathrm{ran}(f)$ for the range of f. To put it in the set-builder notation:*

$$\mathrm{ran}(f) = \{y | y \in E \text{ and there exists an } x \in D \text{ such that } y = f(x)\}$$

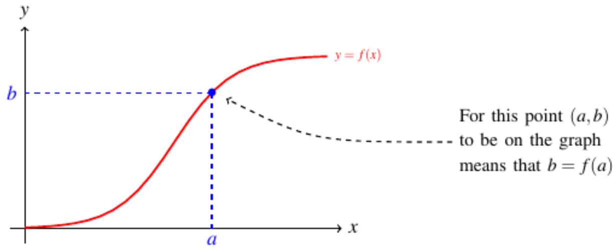
We think of a function as a machine:

The same input x always produces the same output $f(x)$.

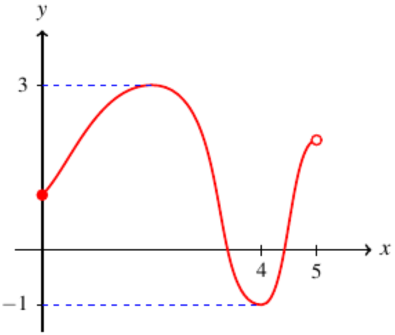
Definition 1.9 (Graphs). *The graph of a function f is the set of all pairs (x, y) of real numbers such that x is in the domain of f and $y = f(x)$. So as a set the graph is*

$$\{(x, f(x)) \mid x \text{ is in the domain of } f\}.$$

We usually plot the graph on the plane.



Question Here is the graph of f.



In this image a solid dot means that the point is on the graph, but a hollow dot means that the point is not on the graph.

1. What is $f(4)$?
2. What is the domain of f?
3. What is the range of f?

Solution

1. -1
2. $[0, 5)$
3. $[-1, 3]$

Example 1.10. *We usually define functions using formulas. For example, let* f *be the function with domain* $D = [0, \infty)$, *with* $E = \mathbb{R}$, *and defined by* $f(x) = x^2$ *for* $x \in D$. *The range of* f *is* $[0, \infty)$ *because* $f(x) = x^2 \geqslant 0$ *for all* $x \in D$, *and every element of* $[0, \infty)$ *has the form* $f(x)$ *for some* $x \in D$.

Definition 1.11 (Domain convention). *If a function is given by a formula, and the domain has not been specified, then our domain convention is that the domain of the function is the set of all numbers for which the formula makes sense and defines a real number. We will always use this domain convention unless otherwise specified.*

Example 1.12. *Let f and g be the functions defined by $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$. The formula $\sqrt{x}$ makes sense and defines a real number so long as $x \geqslant 0$. So the domain convention says that $\operatorname{dom}(f) = \{x \mid x \geqslant 0\} = [0, \infty)$. And the formula $\frac{1}{x}$ makes sense and defines a real number so long as $x \neq 0$. (We can never divide by 0!) So the convention says that $\operatorname{dom}(g) = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.*

Question Find the domain of the functions f and g defined as follows.

1. $f(x) = \sqrt{x+2}$
2. $g(x) = \frac{1}{x^2 - x}$

Write your answer as a union of intervals.

Solution

1. $[-2, \infty)$
2. $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Definition 1.13 (Piecewise defined functions). *A piecewise defined function is one that is defined in different ways on different parts of the domain. We do this by listing the different definitions and the parts of the domain on which they apply, all inside a big brace symbol. For example, we may define a function f with domain $\mathbb{R}$ as follows.*

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } x \geqslant 0 \end{cases}$$

So to work out $f(-1)$ we note that $-1 < 0$, so the first of the two formulas applies, and we get $f(-1) = -1$. And to work out $f(\pi/2)$ we note that $\pi/2 \geqslant 0$, so that the second of the two formulas applies, and we get $f(\pi/2) = \sin(\pi/2) = 1$.

- *Note that the definition covers the whole of the domain $\mathbb{R}$, in the sense that every $x \in \mathbb{R}$ obeys either $x < 0$ or $x \geqslant 0$.*
- *Note that for each x, $f(x)$ is only defined once. In other words, the regions determined by $x < 0$ and $x \geqslant 0$ do not overlap.*
- *There can be two 'pieces' in the piecewise definition, as above, or there can be three or ten or ...*
- *We can sometimes change the defining regions without changing the function. For example, in this case substituting $x = 0$ into either $f(x) = x$ or $f(x) = \sin(x)$ gives $f(x) = 0$ in both cases. This means that we could have defined the two pieces on the regions $x \leqslant 0$ and $x > 0$ without changing f. (So there was a choice, and either one is fine!)*

Example 1.14 (The absolute value function). *The absolute value function sends a real number x to the distance from 0 to x. It is denoted by $|x|$, sometimes pronounced 'mod x' or 'modulus of x'. Distance is never negative, so*

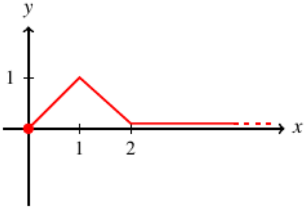
$$|\pi| = \pi, \qquad |-\pi| = \pi, \qquad |-10| = 10, \qquad |0| = 0.$$

In general, the formula is:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Let's check this. To work out $|-\pi|$, we observe that $x = -\pi$ satisfies $x < 0$ and so the second definition $|x| = -x$ applies, giving us $|-\pi| = -(-\pi) = \pi$.

Question Find a formula for the function f.



Solution

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x < 1 \\ 2 - x & \text{if } 1 \leqslant x < 2 \\ 0 & \text{if } 2 \leqslant x \end{cases}$$

Definition 1.15 (Even and odd functions).

- *If a function f satisfies the rule $f(-x) = f(x)$ for every x in its domain, then f is called an even function.*
- *If it satisfies $f(-x) = -f(x)$ for every x in its domain, then it is called an odd function.*

Example 1.16. *Determine whether the function is even or odd.*

1. $f(x) = x^5 + x$
2. $g(x) = 1 - x^4$
3. $h(x) = 2x - x^2$

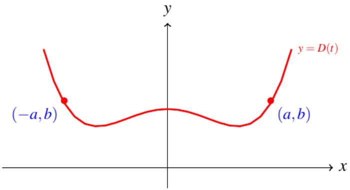
Solution

To find out whether a function f is even or odd, we start by working out $f(-x)$ and simplifying, and we then see whether it is equal to $f(x)$ or $-f(x)$.

1. $f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$ and so f is odd (and not even).
2. $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$ and so g is even (and not odd).
3. $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$. This is not equal to $h(x)$ or to $-h(x)$, and so h is neither even nor odd.

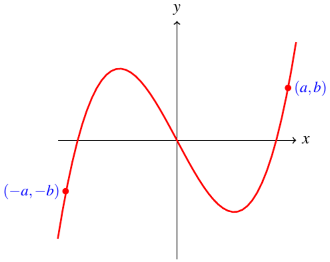
Warning A function does not have to be even or odd. In fact most functions are neither even nor odd. It is even possible for a function to be both even and odd.

It can be helpful to understand odd and even functions in terms of their graphs. The graph of an even function is symmetric under reflection in the y-axis.



Why is this? Our claim is that if (a, b) lies on the graph of f then so does $(-a, b)$. If (a, b) is on the graph then $b = f(a)$, but since f is even then $f(-a) = f(a)$, so that $b = f(-a)$, and consequently $(-a, b)$ also lies on the graph

The graph of an odd function is symmetric under 180° rotation through $(0, 0)$.



Why is this? Our claim is that if (a, b) lies on the graph of f then so does $(-a, -b)$. If (a, b) is on the graph then $b = f(a)$, but since f is odd we have $f(-a) = -f(a)$, so that $-b = -f(a) = f(-a)$, and consequently $(-a, -b)$ also lies on the graph.

Definition 1.17 (Increasing and decreasing functions). *A function f is called increasing on an interval I if the following holds:*

Whenever $x_1, x_2 \in I$ and $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

And it is called decreasing if the following holds:

Whenever $x_1, x_2 \in I$ and $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

Example 1.18.

- *If g is the function defined by $g(x) = x^2$ then g is increasing on $(0, 1)$. That is because if $x_1, x_2 \in (0, 1)$ and $x_1 < x_2$, then $x_1^2 < x_2^2$, so that $g(x_1) < g(x_2)$.*
- *If h is the function defined by $h(x) = -x$, then h is decreasing on $(0, 1)$. That is because if $x_1, x_2 \in (0, 1)$ and $x_1 < x_2$, then $-x_1 > -x_2$, so that $h(x_1) > h(x_2)$.*

In terms of graphs, a function is increasing on an interval I if the graph goes upwards as you move from left to right along I. Similarly, the function is decreasing on an interval I if the graph does downwards as you move from left to right along I.