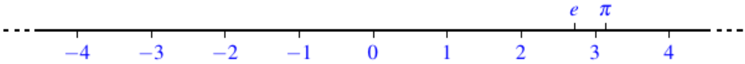
1. Functions

1-1 Sets

Definition 1.1 (Sets and elements). *A set is a collection of objects (maybe numbers, maybe kittens, maybe something else). The objects in the set are called the elements of the set. If S is a set, and a is an element of S, then we write $a \in S$. If a is not an element of S, we write $a \notin S$.*

Example 1.2.

- *$\mathbb{N}$ denotes the set of natural numbers, i.e. whole positive numbers.*
- *$\mathbb{Z}$ denotes the set of integers, i.e. whole numbers, both positive and negative.*
- *$\mathbb{Q}$ denotes the set of rational numbers. These are numbers that we can write as the quotient of two integers, i.e. $2 = \frac{2}{1}, 1.5 = \frac{3}{2}, -2\frac{1}{3} = \frac{-7}{3}$.*
- *$\emptyset$ is the empty set, which is the set with no elements at all.*
- *$\mathbb{R}$ denotes the set of real numbers, i.e. all the numbers on the number line.*



Definition 1.3 (Describing sets). *Some sets can be described by listing their elements between braces, i.e. the symbols* $\{$ *and* $\}$. *For example:*

- $\{0\}$ *is a set with one element,* 0.
- $\{-1, 2, 5\}$ *is a set with three elements.*
- $\{\}$ *is a set with no elements, or in other words it is the empty set* $\emptyset$.
- $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ *is another way of writing* $\mathbb{Z}$.

Sets can also be described using set-builder notation, for example

$$A = \{x \mid x \text{ is an even integer}\} = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}.$$

Definition 1.4 (Intersections and unions). *If S and T are sets, then their intersection, denoted $S \cap T$, is the set of all elements of S that are also elements of T. So in set-builder notation*

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}.$$

Their union, denoted $S \cup T$, is the set consisting of all elements of S and all elements of T. So in set-builder notation:

$$S \cup T = \{x \mid x \in S \text{ or } x \in T \text{ or both}\}.$$

Their set difference, denoted $S \backslash T$ is the set containing all the elements of S which are not elements of T:

$$S \backslash T = \{x | x \in S \text{ and } x \notin T\}.$$

Example 1.5.

$$
\begin{aligned}
\{-2,-1,0\} \cup \{0, 1\frac{1}{2}, \pi\} &= \{-2,-1,0,1\frac{1}{2},\pi\} \\
\{-2,-1,0\} \cap \{0, 1\frac{1}{2}, \pi\} &= \{0\} \\
\{-2,-1,0\} \cap \{1,2\} &= \emptyset \\
\{-2,-1,0\} \cup \{1,2\} &= \{-2,-1,0,1,2\} \\
\{-2,-1,0\} \backslash \{1,2\} &= \{-2,-1,0\} \\
\{-2,-1,0\} \backslash \{0,-1,\frac{1}{2},\pi\} &= \{-2\}
\end{aligned}
$$

Warning The elements of a set are not ordered, so that for example $\{1,2\} = \{2,1\}$. And we can list the same element twice without changing the set, so that for example $\{1,2,2,3,3,3,3,3\} = \{1,2,3\}$.

Definition 1.6 (Intervals). *Let $a < b$ be real numbers.*

- *The open interval (a, b) denotes the set of all numbers between a and b, not including a or b. So in set-builder notation:*

$$(a, b) = \{x \mid a < x < b\}$$

- *The closed interval $[a, b]$ denotes the set of all numbers between a and b, including a and b themselves. So in set-builder notation:*

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}$$

- *There are also two kinds of half-open interval defined as follows:*

$$[a, b) = \{x \mid a \leqslant x < b\}$$
$$(a, b] = \{x \mid a < x \leqslant b\}$$

Question

Describe the following sets as a single interval.

1. $(1, 3) \cup (2, 4]$
2. $(\pi, 10) \cup [10, 11]$
3. $(0, 1) \cup (1, 2)$

Solution

1. $(1, 4]$
2. $(\pi, 11]$
3. It's not possible as single interval, but could write $(0, 2)\backslash\{1\}$

Definition 1.7 (Unbounded Intervals). *There are also unbounded intervals defined as follows.*

$$
\begin{aligned}
(a, \infty) &= \{x \mid a < x\} \\
(-\infty, b) &= \{x \mid x < b\} \\
[a, \infty) &= \{x \mid a \leqslant x\} \\
(-\infty, b] &= \{x \mid x \leqslant b\} \\
(-\infty, \infty) &= \mathbb{R}
\end{aligned}
$$

Warning: *Note that the symbol ∞ only appears in the names on the left, but not as a number on the right. In fact, ∞* **is not a number**, *just a symbol we use for abbreviation.*