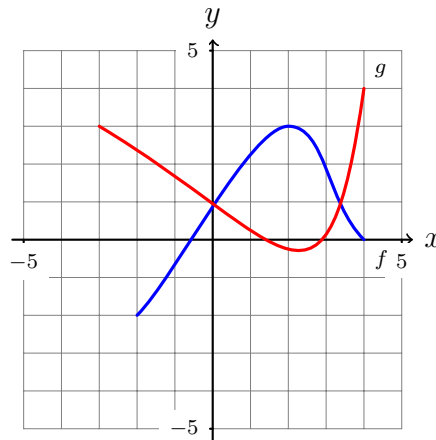


1.2 Functions

1. The graphs of functions f and g are as shown.



- (a) Estimate the solution of the equation $f(x) = -1$.
- (b) State the domain and range of f .
- (c) State the domain and range of g .
2. Find the domain of the function.
- (a) $f(x) = \frac{x+9}{x^2-4}$ (b) $g(x) = \frac{x+9}{x^2+4}$
- (c) $h(x) = \frac{5x^3-2}{x^2+x-2}$ (d) $i(t) = \sqrt{2-t} + \sqrt{2+t}$
- (e) $j(u) = \frac{1}{\sqrt[4]{u^2-6u}}$ (f) $k(u) = \frac{1}{\sqrt[3]{u^2-6u}}$
- (g) $\ell(t) = \sqrt{4-\sqrt{1+t}}$ (h) $m(t) = \frac{\sqrt{4-\sqrt{1+t}}}{t}$

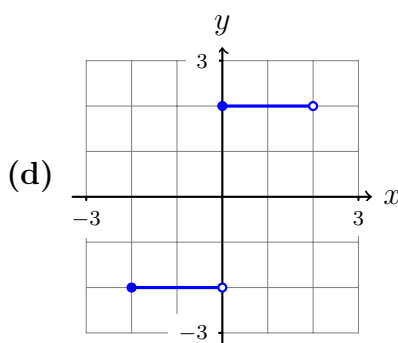
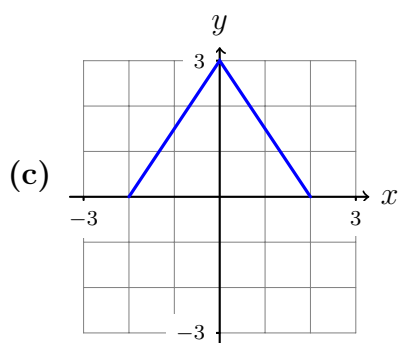
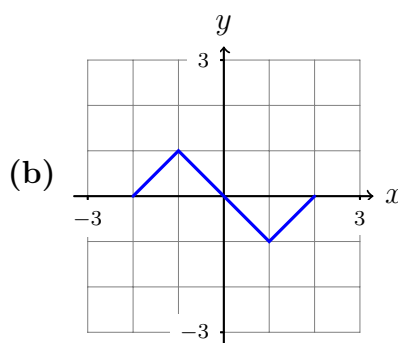
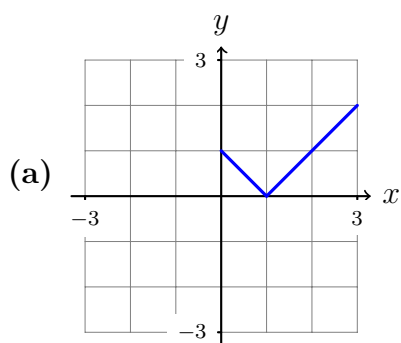
3. Sketch the graph of the given function.

(a) $g(x) = 0.5(|x| + x)$ (b) $f(x) = \begin{cases} -x + 3 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$

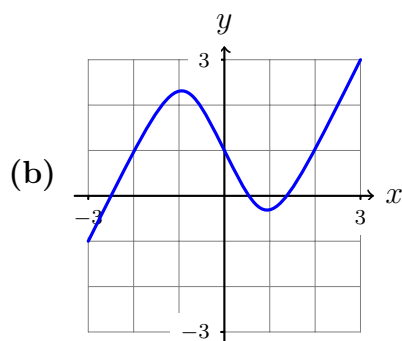
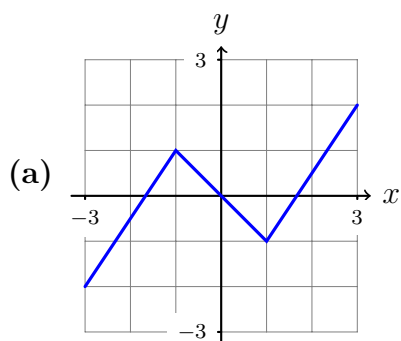
(c) $h(x) = \begin{cases} 4 - 2x & \text{if } x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$ (d) $k(x) = \begin{cases} 4 - 2x & \text{if } x < 2 \\ x - 3 & \text{if } x \geq 2 \end{cases}$

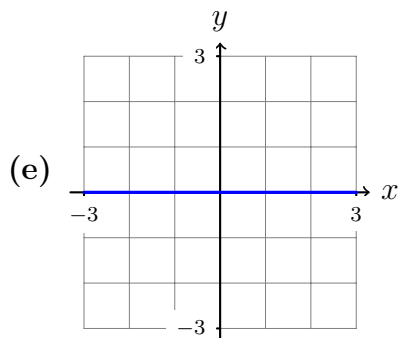
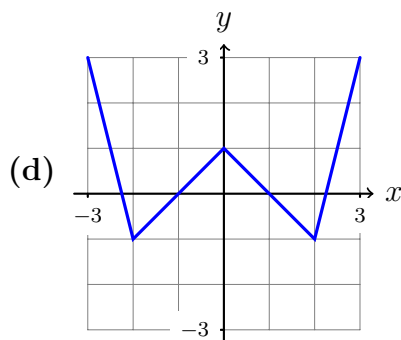
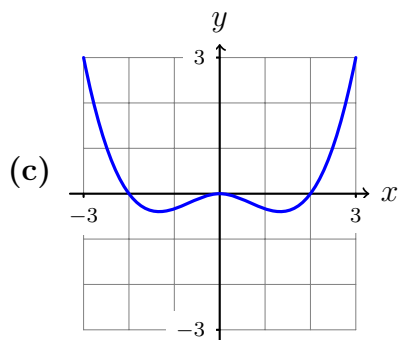
(e) $f(x) = \begin{cases} -x - 6 & \text{if } x < -2 \\ x^3 & \text{if } x \geq -2 \end{cases}$ (f) $f(x) = \begin{cases} x^2 & \text{if } |x| \geq 1 \\ |x| & \text{if } |x| < 1 \end{cases}$

4. Find expressions for the given functions, and state their ranges.



5. For each function, say whether it is even or odd (or neither or both).





6. Is each function even or odd (or neither or both)?

(a) $f(x) = \frac{x^3}{x^3 + x}$

(b) $h(x) = \frac{x^3}{x^2 + 1}$

(c) $h(x) = \frac{x^3}{x^3 + 1}$

(d) $f(x) = \frac{x^2}{x^3 + x}$

(e) $g(t) = 3t^2 - 5$

(f) $f(u) = -u - 5$

(g) $f(x) = 3x^3 - x$

(h) $g(x) = |x| \cdot x^2$

(i) $h(t) = \frac{|t| - t^4}{t - t^3}$

Now let p and q be even functions, and let r and s be odd functions. Determine whether the following functions are even, or odd, or neither, or both.

(j) $p + q$

(k) $r - s$

(l) pr

(m) $q \circ r$

(n) $p \circ q$

(o) $r \circ s$

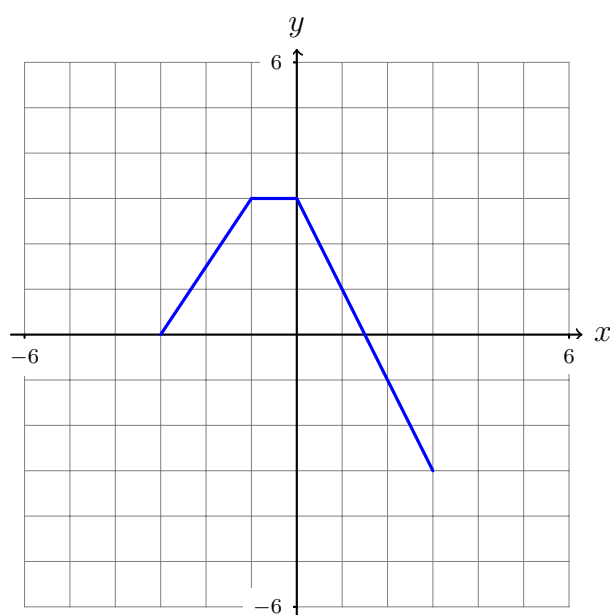
7. Let f be a function, and let functions g and h be defined by

$$g(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad h(x) = \frac{f(x) - f(-x)}{2}.$$

Show that g is even and that h is odd. Conclude that every function is the sum of an even function and an odd function.

1.3 New functions from old functions

8. Here is the graph of a function f .



Draw the graphs of the following functions.

(a) $y = f(x) - 2$

(b) $y = f(x + 2)$

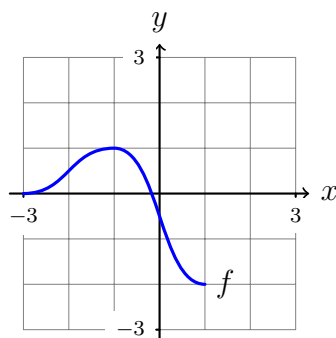
(c) $y = f(3x)$

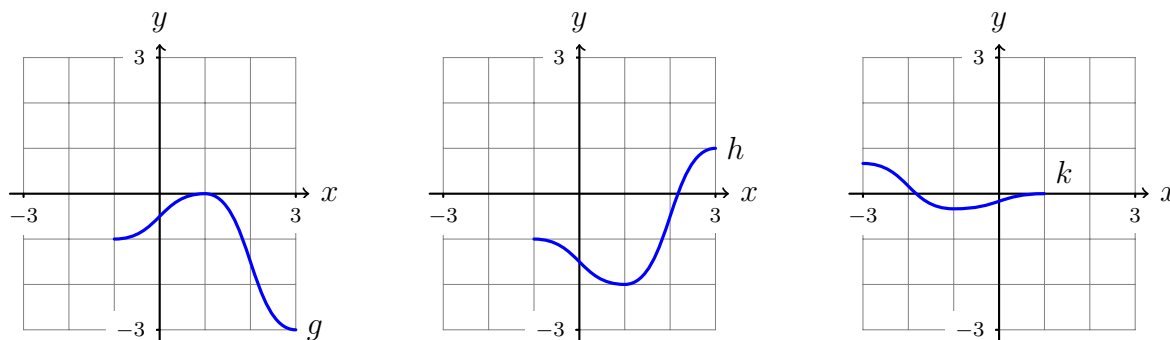
(d) $y = f\left(\frac{1}{2}x\right)$

(e) $y = \frac{1}{3}f(x)$

(f) $y = -f(-x)$

9. The graph of a function f is given. Use transformations to write down functions g , h and k whose graphs are as shown. (Your answers should express the new function in terms of $f(x)$.)



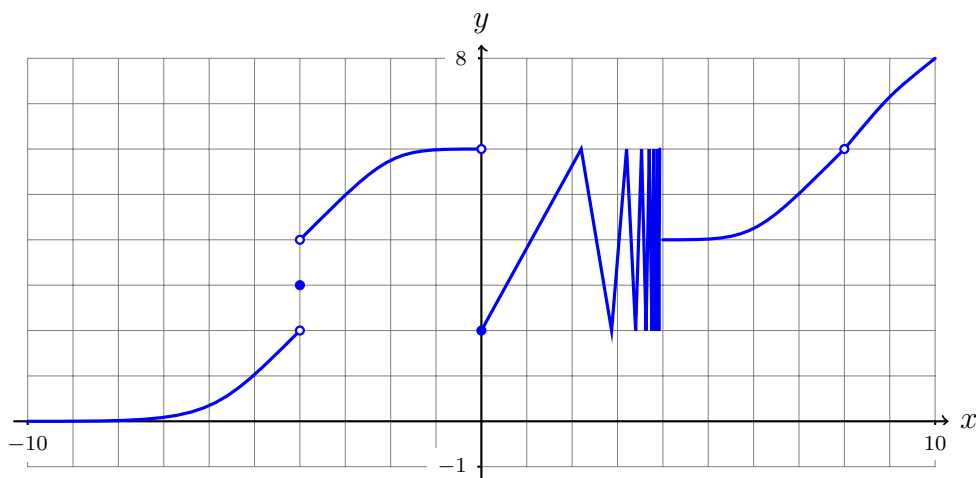


10. Let functions f and g be defined by $f(x) = x^3 + 2$ and $g(x) = x^2 - 4$. Find the following functions and state their domains.
 (a) $f + g$ (b) $f - g$ (c) fg (d) f/g
11. Let functions f and g be defined by $f(x) = \sqrt{5-x}$ and $g(x) = \sqrt{x^2-4}$. Find the following functions and state their domains.
 (a) $f + g$ (b) $f - g$ (c) fg (d) f/g
12. Let functions f and g be defined by $f(x) = 3x+2$ and $g(x) = x^2-1$. Find the following functions and state their domains.
 (a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g$ (d) $f \circ f$
13. Let functions f and g be defined by $f(x) = \sin x$ and $g(x) = 2x+2$. Find the following functions and state their domains.
 (a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g$ (d) $f \circ f$
14. Let functions f and g be defined by $f(x) = 1/x$ and $g(x) = 1/(x+2)$. Find the following functions and state their domains.
 (a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g$ (d) $f \circ f$

1.4 The limit of a function

15. Use the given graph of f to state the value of each quantity, if it exists. If it does not

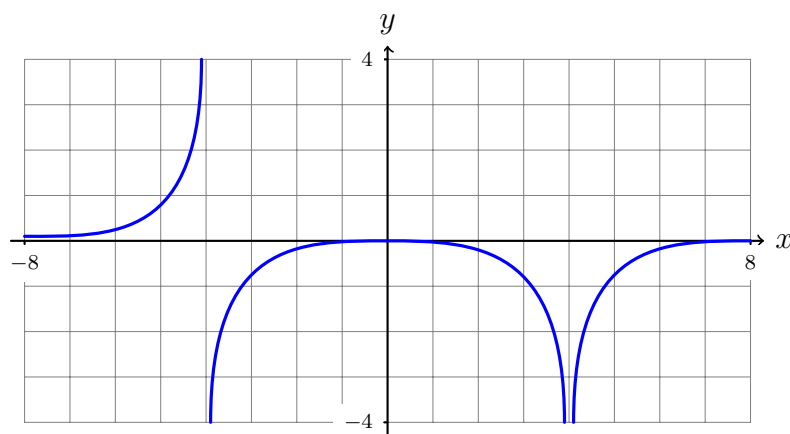
exist, explain why.



- (a) $f(8)$ (b) $\lim_{x \rightarrow 4^-} f(x)$ (c) $\lim_{x \rightarrow 4^+} f(x)$
- (d) $\lim_{x \rightarrow 4} f(x)$ (e) $\lim_{x \rightarrow 0^+} f(x)$ (f) $\lim_{x \rightarrow 0^-} f(x)$
- (g) $\lim_{x \rightarrow 0} f(x)$ (h) $f(0)$ (i) $\lim_{x \rightarrow -4^-} f(x)$
- (j) $\lim_{x \rightarrow -4^+} f(x)$ (k) $f(-4)$ (l) $\lim_{x \rightarrow 8} f(x)$

16. For the function f whose graph is shown, state the following.

- (a) $\lim_{x \rightarrow -4^-} f(x)$ (b) $\lim_{x \rightarrow -4^+} f(x)$ (c) $\lim_{x \rightarrow -4} f(x)$
- (d) $\lim_{x \rightarrow 4^-} f(x)$ (e) $\lim_{x \rightarrow 4^+} f(x)$ (f) $\lim_{x \rightarrow 4} f(x)$



17. Let h denote the function defined as follows.

$$h(x) = \begin{cases} 2 + x, & x < -1 \\ x^2, & -1 \leq x < 1 \\ 1 + x, & 1 \leq x \end{cases}$$

Determine whether the given limits exist.

$$(a) \lim_{x \rightarrow -1} h(x) \qquad (b) \lim_{x \rightarrow 1} h(x)$$

Now, for each function, determine the values of a for which $\lim_{x \rightarrow a} f(x)$ exists.

$$(c) f(x) = \begin{cases} -2 - x, & x < -1 \\ x^3, & -1 \leq x < 1 \\ 1 - x, & 1 \leq x \end{cases}$$

$$(d) f(x) = \begin{cases} \sin x & x \leq 0 \\ 1 + \cos x, & 0 < x < \pi \\ \sin x, & \pi \leq x \end{cases}$$

18. Sketch the graph of an example of a function f that satisfies all of the conditions.

$$(a) \lim_{x \rightarrow 0^-} f(x) = 1, \lim_{x \rightarrow 0^+} f(x) = -1, f(0) = -1.$$

$$(b) \lim_{x \rightarrow 0^-} f(x) = 1, \lim_{x \rightarrow 0^+} f(x) = 0, f(0) \text{ is undefined.}$$

$$(c) \lim_{x \rightarrow 0} f(x) = 1, f(0) = 0, \lim_{x \rightarrow 2^-} f(x) = 1, \lim_{x \rightarrow 2^+} f(x) = 0.$$

$$(d) \lim_{x \rightarrow 0^-} f(x) \text{ does not exist, } \lim_{x \rightarrow 0^+} f(x) = 1, f(0) = 1.$$

19. Determine the infinite limit.

$$(a) \lim_{x \rightarrow \pi/2^-} \tan x \quad (b) \lim_{x \rightarrow \pi/2^+} \tan x \quad (c) \lim_{x \rightarrow 3^+} \frac{1+x}{x-3}$$

$$(d) \lim_{x \rightarrow 3^-} \frac{1+x}{x-3} \quad (e) \lim_{x \rightarrow 0} \frac{1+x}{x^2(x-3)} \quad (f) \lim_{x \rightarrow 3^+} \frac{1+x}{x^2(x-3)}$$

20. Given that $\lim_{x \rightarrow 1} f(x) = 2$, $\lim_{x \rightarrow 1} g(x) = 0$ and $\lim_{x \rightarrow 1} h(x) = -3$, find the limits that exist. Justify each step. If a limit does not exist, explain why.

$$(a) \lim_{x \rightarrow 1} [3f(x) + 2g(x)]$$

$$(b) \lim_{x \rightarrow 1} [f(x)]^4$$

$$(c) \lim_{x \rightarrow 1} \sqrt[3]{h(x)}$$

$$(d) \lim_{x \rightarrow 1} \frac{f(x)}{4h(x)}$$

$$(e) \lim_{x \rightarrow 1} \frac{g(x)f(x)}{h(x)}$$

21. Evaluate the following limits and justify each step.

$$(a) \lim_{x \rightarrow 2} (3x^4 - 2x^3 + x - 3)$$

$$(b) \lim_{x \rightarrow -2} (x^3 + 3)(x^2 + 4x + 1)$$

(c) $\lim_{u \rightarrow 3} \sqrt{u^3 - 3u + 3}$

(d) $\lim_{t \rightarrow 1} \sqrt{\frac{t^2 + t + 1}{2t + 3}}$

22. What is wrong with the following equation?

$$\frac{x^2 - 5x + 6}{x - 3} = x - 2$$

With this in mind, explain why the equation

$$\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \rightarrow 3} (x - 2)$$

is true.

23. Compute the following limits, or explain why they do not exist.

(a) $\lim_{x \rightarrow -1} \frac{x^2 + 3x + 2}{x + 1}$

(b) $\lim_{x \rightarrow -1} \frac{x^2 + 3x + 1}{x + 1}$

(c) $\lim_{x \rightarrow 3} \frac{2x^2 - 5x - 3}{x^2 - 2x - 3}$

(d) $\lim_{x \rightarrow -2} \frac{x^2 + x - 6}{x^2 - 4}$

(e) $\lim_{h \rightarrow 0} \frac{(h - 2)^2 - 4}{h}$

(f) $\lim_{h \rightarrow 0} \frac{(h - 2)^3 + 8}{h}$

(g) $\lim_{h \rightarrow 0} \frac{(h - 2)^3 + 8}{h^2}$

(h) $\lim_{t \rightarrow 0} \frac{\sqrt{2+t} - \sqrt{2-t}}{t}$

(i) $\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{2}{t^2 + 2t} \right)$

(j) $\lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{9x - x^2}$

24. (a) Show that $\lim_{x \rightarrow 0} |x| = 0$.

(b) Let f be the function defined by

$$f(x) = \begin{cases} x \cos(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that $-|x| \leq f(x) \leq |x|$ for all x .

(c) Use the squeeze theorem to show that $\lim_{x \rightarrow 0} f(x) = 0$.

25. (a) If $x^2 - 5x + 8 \leq f(x) \leq 2x^2 - 11x + 17$, find $\lim_{x \rightarrow 3} f(x)$.

(b) If $x^2 + 2x + 2 \leq f(x) \leq 2x^2 + 4x + 3$, find $\lim_{x \rightarrow -1} f(x)$.

26. Evaluate the limit if it exists. If it does not, explain why.

$$(a) \lim_{x \rightarrow -3} \frac{3x + 9}{|x + 3|} \quad (b) \lim_{x \rightarrow 3} \frac{3x + 9}{|x + 3|} \quad (c) \lim_{x \rightarrow -3} \frac{|x| - 3}{x + 3}$$

27. Let a , b and c be constants. Let p be the function defined by $p(x) = ax^2 + bx + c$. Using *only* the limit laws 1, 3, 4, 7 and 8, show that $\lim_{x \rightarrow a} p(x) = p(a)$.

28. Suppose that $\lim_{x \rightarrow 0} \frac{g(x)}{x^3} = 4$. Find $\lim_{x \rightarrow 0} \frac{g(x)}{x^2}$ and $\lim_{x \rightarrow 0} \frac{g(x)}{x}$.

29. (a) Give examples of functions f and g such that $\lim_{x \rightarrow 0} [f(x) + g(x)]$ exists but $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not.

(b) Give examples of functions f and g such that $\lim_{x \rightarrow 0} [f(x)/g(x)]$ exists but $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not.

1.5 The precise definition of a limit

30. Read the following example and its solution, which is not correct. Describe how to fix the solution.

Example: Show that $\lim_{x \rightarrow 2} (2x + 3) = 7$.

Solution: Given $\epsilon > 0$, define $\delta = \epsilon$. Then if $|x - 2| < \delta$, we have

$$\begin{aligned} |(2x + 3) - 7| &= |2x - 4| \\ &= |2(x - 2)| \\ &= |2| \cdot |x - 2| \\ &= 2|x - 2| \\ &< 2\delta \\ &= 2\epsilon \end{aligned}$$

so that $|(2x + 3) - 7| < 2\epsilon$ as required.

31. Read the following example and its solution, which is not correct. Describe how to fix the solution.

Example: Show that $\lim_{x \rightarrow 3} (17 - 5x) = 2$.

Solution: Given $\epsilon > 0$, define $\delta = \epsilon/5$. Then if $|x - 3| < \delta$, it follows that

$$\begin{aligned} |(17 - 5x) - 2| &= |15 - 5x| \\ &= |(-5)(3 - x)| \\ &= -5|3 - x| \\ &= 5|x - 3| \\ &< 5\delta \\ &= 5\epsilon/5 \\ &= \epsilon \end{aligned}$$

so that $|(17 - 5x) - 2| < \epsilon$ as required.

32. Use the **definition** of limit to prove the following.

$$\text{(a) } \lim_{x \rightarrow 1} (2x + 2) = 4. \quad \text{(b) } \lim_{x \rightarrow 1} (3 - 2x) = 1. \quad \text{(c) } \lim_{x \rightarrow 2} \left(\frac{1}{2}x - 3 \right) = -2.$$

33. Use the **definition** of limit to prove the following.

$$\begin{aligned} \text{(a) } \lim_{x \rightarrow a} x &= a. & \text{(b) } \lim_{x \rightarrow a} c &= c. & \text{(c) } \lim_{x \rightarrow 0} |x| &= 0. \\ \text{(d) } \lim_{x \rightarrow 0} \sqrt[3]{x} &= 0. & \text{(e) } \lim_{x \rightarrow 1} \frac{1}{x} &= 1. & \text{(f) } \lim_{x \rightarrow -5} x^2 &= 25. \end{aligned}$$

34. Read the following example and its solution, then answer the questions that follow.

Example: Show that $\lim_{x \rightarrow 2} \left(\frac{1}{x+1} \right) = \frac{1}{3}$.

Solution: Given $\epsilon > 0$, define $\delta = \min(2, 3\epsilon)$. Suppose that $0 < |x - 2| < \delta$.

Since $|x - 2| < \delta$ and $\delta \leq 2$, we have $|x - 2| < 2$. It follows that $-2 < x - 2 < 2$. By adding 3 to all terms we find that $1 < x + 1 < 5$. Consequently $1 < |x + 1|$, and rearranging gives $\frac{1}{|x+1|} < 1$.

Now

$$\begin{aligned} \left| \frac{1}{x+1} - \frac{1}{3} \right| &= \left| \frac{3 - (x+1)}{3(x+1)} \right| \\ &= \left| \frac{2-x}{3(x+1)} \right| \\ &= \frac{1}{3} \cdot \frac{1}{|x+1|} \cdot |2-x| \\ &= \frac{1}{3} \cdot \frac{1}{|x+1|} \cdot |x-2| \\ &< \frac{1}{3} \cdot 1 \cdot \delta \\ &\leq \frac{1}{3} \cdot 1 \cdot 3\epsilon \\ &= \epsilon \end{aligned}$$

and so $\left| \frac{1}{x+1} - \frac{1}{3} \right| < \epsilon$. Thus $\lim_{x \rightarrow 2} \left(\frac{1}{x+1} \right) = \frac{1}{3}$ as required. \square

(a) Suppose that a and b are real numbers, and set $\delta = \min(a, b)$. Explain why the inequalities $\delta \leq a$ and $\delta \leq b$ are true. Where are these inequalities used in the solution above?

(b) Where is the identity $|-a| = |a|$ used in the solution above?

(c) The second paragraph of the solution shows that if $|x - 2| < 2$ then $\frac{1}{|x+1|} < 1$. If we assume instead that $|x - 2| < 1$, then what could we conclude about $\frac{1}{|x+1|}$? What if we assume that $|x - 2| < 3$?

(d) Write a new version of the solution where δ has the form $\min(1, b)$ for an appropriate choice of b . (You will need to adjust the solution in several other places to ensure that it still works.)

(e) What happens if you try to write a new version of the solution where δ has the form $\min(3, b)$?

35. Use the **definition** of limit to prove the following.

(a) $\lim_{x \rightarrow 2} (x^2 - x - 3) = -1$

(b) $\lim_{x \rightarrow -1} (x^2 + 4x + 4) = 1$

(c) $\lim_{x \rightarrow 1} (2x^2 - 2x + 3) = 3$

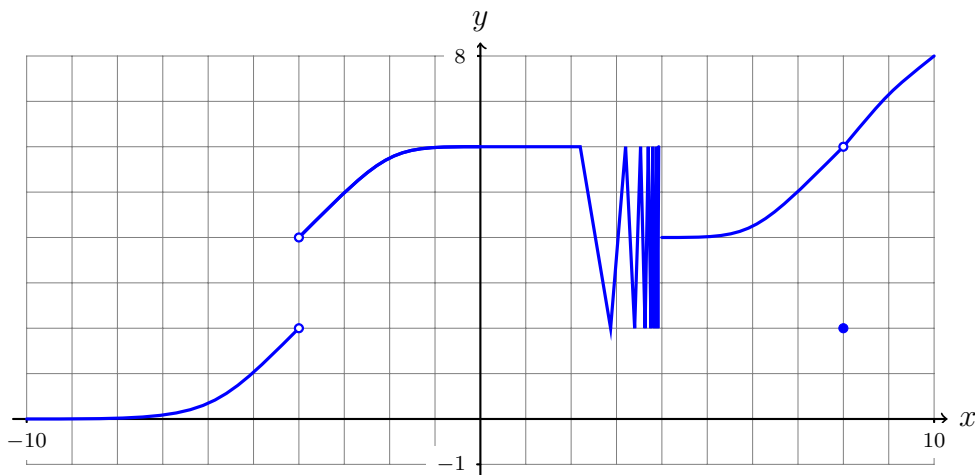
(d) $\lim_{x \rightarrow 1} \frac{x - 2}{x + 3} = -\frac{1}{4}$

(e) $\lim_{x \rightarrow 4} \frac{2x + 1}{x - 2} = \frac{9}{2}$

(f) $\lim_{x \rightarrow -2} \frac{3x - 1}{2x + 1} = \frac{7}{3}$

1.6 Continuity

36. Let f be the function whose graph is as shown.



(a) Explain why $a = -4$ is a discontinuity of f .

(b) Explain why $a = 4$ is a discontinuity of f .

(c) Explain why $a = 8$ is a discontinuity of f .

(d) On which intervals is f continuous?

37. In each case, sketch the graph of a function that is continuous except for the stated discontinuities.

(a) Discontinuous at 0, but continuous from the left there.

(b) Discontinuous at -1 and 1 , but continuous from the left at 1 and from the right at -1 .

(c) Continuous from the right, but not from the left, at every integer.

38. Explain why f is discontinuous at a .

(a) $a = 2$ and $f(x) = \begin{cases} x^2 - 2x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(b) $a = 0$ and $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(c) $a = 0$ and $f(x) = \begin{cases} \sin x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } x > 0 \end{cases}$

39. At which numbers is f discontinuous? At which of these is f continuous from the right or left or neither?

(a) $f(x) = \begin{cases} \sin(1/x) & \text{if } x < 0 \\ 0 & \text{if } 0 < x \leq 1 \\ x^2 - 1 & \text{if } 1 < x \end{cases}$

(b) $f(x) = \begin{cases} \sin x & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ x^2 - 1 & \text{if } 1 < x \end{cases}$

(c) $f(x) = \begin{cases} \sin(1/x) & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \\ x^2 - 1 & \text{if } 1 < x \end{cases}$

(d) $f(x) = \begin{cases} \sin x & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \\ x^2 & \text{if } 1 < x \end{cases}$

40. (a) Let f be the function defined as follows.

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ cx^2 - x & \text{if } 1 \leq x \end{cases}$$

For which value of c is f continuous at $a = 1$?

(b) For what value of the constant b is the function f defined by

$$f(x) = \begin{cases} 2x^2 + bx & \text{if } x < -1 \\ 2x + b & \text{if } -1 \leq x \end{cases}$$

continuous on $(-\infty, \infty)$?

41. Use the intermediate value theorem to show that:

(a) There is $c \in (0, 1)$ such that $c^2 + 2c - 2 = 0$.

(b) There is $c \in (-\pi/2, \pi/2)$ such that $\sin(c) - 2\cos(c) = 0$.

(c) There is $x \in (0, 1)$ such that $\sqrt[5]{x} = 1 - x$.

2.1 Derivatives and rates of change

42. Sketch the graph of a function f for which $f(0) = 2$, $f'(-2) = 1$, $f'(2) = 0$, $\lim_{x \rightarrow 4^-} f(x) = \infty$, $\lim_{x \rightarrow -4^+} f(x) = -\infty$.
43. Let a be any real number. Compute $f'(a)$ using the definition of the derivative.
- (a) $f(x) = c$ (b) $f(x) = x$ (c) $f(x) = x^2$ (d) $f(x) = \frac{1}{x}$
44. Let F be the function defined by $F(x) = \frac{3}{x^2+x}$.
- (a) Use the definition of the derivative to find $F'(2)$.
- (b) Give the equation of the tangent line to the curve $y = F(x)$ at the point $(2, 1/2)$.
45. Use the definition of the derivative to find $f'(a)$.
- (a) $f(x) = 2x^2 - 3x + 4$ (b) $f(t) = t^3 - 2$ (c) $f(t) = \frac{t+2}{2t+3}$
- (d) $f(x) = \sqrt{2-x}$ (e) $f(x) = \frac{1}{\sqrt{x+1}}$
46. Let f be a function, let c be a constant, and define a new function g by $g(x) = f(x) + c$. Use the definition of derivative to show that $g'(a) = f'(a)$.

2.2 The derivative as a function

47. State the derivative of the function, the domain of the function, and the domain of the derivative.
- (a) $f(x) = c$ (b) $f(x) = x$ (c) $f(x) = x^2$ (d) $f(x) = \frac{1}{x}$
48. Find the derivative of the function using the definition of derivative. State the domain of each function and the domain of the derivative.
- (a) $f(x) = x - \sqrt{x}$ (b) $g(x) = \sqrt{-x}$ (c) $f(x) = \frac{2-3x}{2+3x}$
49. Use the definition of derivative to determine whether $f'(0)$ exists.
- (a) $f(x) = x^{\frac{3}{5}}$ (b) $f(x) = x^{\frac{5}{3}}$ (c) $f(x) = |x|^2$
50. (a) Sketch the graph of a function which is continuous on $(-\infty, \infty)$ and differentiable at every real number except the integers.
- (b) Sketch the graph of a function that is continuous on $(-1, 1)$, differentiable on $(-1, 1)$, and such that $\lim_{x \rightarrow -1} f'(x) = \infty$, $\lim_{x \rightarrow 1} f'(x) = \infty$, and $f'(x) > 0$ for all $x \in (-1, 1)$.

51. Suppose that f is an even function. What can you conclude about f' ? Prove your answer.

2.3 Differentiation formulas

52. Express the following in the form x^c for an appropriate choice of c .

(a) $\sqrt[n]{x}$ (b) $(x^a)^b$ (c) $x^a x^b$ (d) $\frac{1}{x^a}$ (e) $\sqrt[3]{x} \sqrt[4]{x}$ (f) $\frac{1}{x^{5/2} \sqrt[3]{x}}$

53. Differentiate the function using differentiation formulas.

(a) $f(x) = 22.5$ (b) $f(x) = \sqrt{50}$ (c) $f(x) = 3x + 2$
 (d) $F(x) = -7x^{11}$ (e) $f(x) = 2x^3 - x^2 - 1$ (f) $f(t) = \frac{1}{3}t^5 - 2t^3 + t^2$
 (g) $g(x) = x^{-3/5}$ (h) $A(y) = cy^{-3}$ (i) $B(s) = \frac{8}{s^3}$
 (j) $F(r) = r^{4/3} - r^{1/3}$ (k) $T(p) = p + 2\sqrt{p}$ (l) $A(r) = \pi r^2$
 (m) $g(u) = \sqrt{3u} - \sqrt{3u}$ (n) $f(u) = \sqrt[3]{u} + 3\sqrt{u^3}$

54. Find the derivative of $f(x) = (2x^2 - 1)(x^2 + x)$ in two ways: by using the Product Rule and by multiplying first. Do the answers agree? Which method is faster?

55. Find the derivative of $f(x) = \frac{2x^3 + \sqrt{x} - x}{x^2}$ in two ways: by using the Quotient Rule and by simplifying first. Do the answers agree? Which method is faster?

56. Differentiate the function.

(a) $g(x) = x^3(2 - x)$. (b) $h(x) = (x + 2)(2x - 3)$ (c) $k(x) = \sqrt[3]{x}(2x + 3)$
 (d) $F(t) = (2t - 1)^2$ (e) $H(x) = (x - x^{-1})^3$ (f) $f(x) = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$
 (g) $F(y) = \left(\frac{1}{y^3} + \frac{2}{y^2}\right)(5y - y^2)$ (h) $G(s) = (3s^2 + s)(s^{-3} + s^{-5})$
 (i) $f(t) = \frac{t^2}{1 - t^3}$ (j) $f(t) = \frac{1 - t^3}{t^2}$ (k) $f(x) = \frac{x + 2}{x^2 + 3x + 2}$
 (l) $K(v) = \frac{v^2 - 3v\sqrt[3]{v}}{v}$ (m) $g(t) = \frac{t + t^{1/3}}{\sqrt{t}}$ (n) $f(t) = \frac{t - 4}{\sqrt{t} - 2}$
 (o) $f(x) = \frac{x^2}{x - \frac{1}{x}}$ (p) $f(x) = ax^2 + bx + c$

57. Find the equation of the tangent line to the curve at the given point.

(a) $y = \frac{x}{2x - 1}$, $(1, 1)$. (b) $y = x^3 + 2x^2 + 1$, $(1, 4)$.

58. Find the first and second derivatives of the function.

(a) $f(x) = x^4 - 2x^3 + 3x^2 - 4x + 5$

(b) $G(r) = \sqrt[4]{r} - 2\sqrt[3]{r}$

(c) $f(x) = \frac{x^3}{2+x}$

(d) $f(x) = \frac{1}{2-3x}$

(e) $f(x) = \frac{1}{x^2 - x - 1}$

59. Suppose that $f(2) = 3$, $f'(2) = -1$, $g(2) = 2$ and $g'(2) = 0$. Find $h'(2)$ where h is defined as follows.

(a) $h(x) = 2g(x) - 3f(x)$

(b) $h(x) = f(x)g(x)$

(c) $h(x) = g(x)/f(x)$

(d) $h(x) = f(x)/g(x)$

(e) $h(x) = \frac{f(x)}{1-g(x)}$

Now suppose that p , q and r are differentiable functions satisfying the following.

$$p(1) = 2, \quad q(2) = 3, \quad r(1) = 2, \quad p'(3) = 4, \quad q'(2) = 9, \quad r'(1) = 4$$

(f) The chain rule states that $(q \circ r)'(x) = q'(r(x)) \cdot r'(x)$. Use it to compute $(q \circ r)'(1)$.

(g) Use the chain rule to show that $(p \circ q \circ r)'(x) = p'(q(r(x))) \cdot q'(r(x)) \cdot r'(x)$. Then compute $(p \circ q \circ r)'(1)$.

60. Suppose that g is a differentiable function. Find expressions for the derivatives of the following functions in terms of $g(x)$ and $g'(x)$.

(a) $x^2g(x)$

(b) $\frac{x^2}{g(x)}$

(c) $\frac{g(x)}{x^2}$

(d) $\sqrt{x} + g(x)$

(e) $\frac{g(x)}{1 + \sqrt{x}}$

(f) $x^2 - xg(x)$

61. Exactly one of the tangent lines to the curve $f(x) = \sqrt[3]{x}$ has gradient 3. Find its equation.

62. (a) Show that $\frac{d}{dx} [f(x)]^2 = 2f(x)f'(x)$.

(b) Show that $(fgh)' = f'gh + fg'h + fgh'$.

63. For what values of a is the function f differentiable at a ? You should justify your answer using the definition of the derivative.

(a) $f(x) = |x - 1| + |x + 1|$

(b) $f(x) = \begin{cases} x & \text{if } x < 0 \\ x + x^3 & \text{if } 0 \leq x < 1 \\ x^2 + x^3 & \text{if } 1 \leq x \end{cases}$

2.4 Derivatives of trigonometric functions

64. Remember the special limits

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1, \quad \lim_{\theta \rightarrow 0} \frac{\cos(\theta) - 1}{\theta} = 0.$$

Use the first one to compute the following.

$$(a) \lim_{\theta \rightarrow 0} \frac{\theta}{\sin(\theta)} \quad (b) \lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin(\theta)} \quad (c) \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta + \sin(\theta)} \quad (d) \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta + \theta^2}$$

65. Differentiate the function. Recall that $\csc(x) = \frac{1}{\sin(x)}$ and $\sec(x) = \frac{1}{\cos(x)}$.

$$(a) f(x) = 2 \sin x - x^3 \quad (b) f(x) = x \cos x \quad (c) f(x) = \cos x + \frac{1}{2} \csc x$$

$$(d) y = 2 \csc x - \sec x \quad (e) g(t) = t^2 \sin t \quad (f) g(t) = \tan t - 2 \sec t$$

$$(g) y = a \sin t + t^2 \cos t \quad (h) y = u(a \sin u + b \tan u) \quad (i) y = \frac{x}{2 - \cot x}$$

$$(j) y = \sec x \cdot \tan x \quad (k) f(\theta) = \frac{\tan \theta}{1 + \tan \theta} \quad (l) y = \frac{\cos x}{1 + \sin x}$$

$$(m) y = \frac{1 + \sec x}{\tan x} \quad (n) y = x^2 \cos(x) \cot(x)$$

66. Use the definition of the derivative to prove that $\frac{d}{dx} \cos x = -\sin x$. You will need to use the formula $\cos(a + b) = \cos a \cos b - \sin a \sin b$, and the results $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$, $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$ that we proved in the course.

67. Prove the following.

$$(a) \frac{d}{dx} \csc x = -\csc x \cdot \cot x \quad (b) \frac{d}{dx} \sec x = \sec x \cdot \tan x \quad (c) \frac{d}{dx} \cot x = -\csc^2 x.$$

68. Find the equation of the tangent line to the given curve through the given point.

$$(a) y = \sec x, \left(\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right) \quad (b) y = (1 - x) \sin x, (0, 0) \quad (c) y = x + \sec x, (\pi, \pi - 1)$$

69. Compute $\frac{d^{45}}{dx^{45}}[\sin x]$ and $\frac{d^{16}}{dx^{16}}[x \cos x]$.

70. For what values of the x does the curve $y = \frac{\sin x}{2 + \cos x}$ have horizontal tangent?

77. Use implicit differentiation to find an equation of the tangent line to the curve through the given point.

(a) $y \cos(2x) = x \sin(2y)$, $(\pi/4, \pi/2)$

(b) $\sin(x - y) = 2x + 2y$, $(\pi/2, -\pi/2)$

(c) $x^2 - xy + y^2 = 3$, $(1, -1)$

(d) $x^{2/3} + y^{2/3} = 4$, $(1, 3\sqrt{3})$

(e) $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(\sqrt{5}, 2)$

78. Find y'' by implicit differentiation.

(a) $x^2 + 4y^2 = 4$

(b) $\sqrt{x} - \sqrt{y} = 1$

(c) $x^5 + y^5 = 1$

(d) $xy - y^3 = 1$

2.7 Maximum and minimum values

79. For each part, sketch the graph of a function f on $[0, 2]$ with the stated property.

(a) f has a local maximum at 1 and is differentiable at 1.

(b) f has a local maximum at 1 and is continuous but not differentiable at 1.

(c) f has a local maximum at 1 and is not continuous at 1.

(d) f has a local maximum at 1 but not an absolute maximum at 1.

80. Find the critical numbers.

(a) $f(x) = 6x^2 + 3x$

(b) $f(x) = x^3 - 3x^2 + 3x$

(c) $g(t) = |4t - 3|$

(d) $g(y) = \frac{y + 1}{y^2 + y + 1}$

(e) $h(t) = 2t^{3/4} - t^{1/4}$

(f) $F(x) = x^{3/5}(x - 3)^3$

81. Find the absolute maximum and minimum of the function in the given interval. In your solution you must state what method you are using, and you must justify why it can be applied.

(a) $f(x) = 2x^2 - 4x + 3$, $[0, 3]$

(b) $f(x) = x^3 - 12x + 2$, $[0, 3]$.

(c) $f(x) = 3x^4 + 12x^3 + 12x^2 - 5$, $[0, 2]$.

(d) $f(t) = t\sqrt{9 - t^2}$, $[-1, 3]$.

(e) $f(t) = 2 \sin t - \cos 2t$, $[0, 2\pi]$.

2.8 The Mean Value Theorem

82. Verify that the function satisfies the hypotheses of the mean value theorem on the interval $[0, 2]$, and then find all numbers c that satisfy the conclusion of the mean value theorem.
- (a) $f(x) = 2x^2 + 3x - 1$ (b) $f(x) = \sqrt{x}$
83. Show that the equation $2x + \sin(x) = 0$ has exactly one real root. (In other words, show that there is exactly one number x_0 for which $2x_0 + \sin(x_0) = 0$.)
84. (a) Show that the equation $x^3 - 14x + 5 = 0$ has at most one root in the interval $[-2, 2]$.
 (b) Show that the equation $x^4 + 5x^2 + 2 = 0$ has at most two real roots.
85. (a) Suppose that f is a differentiable function satisfying $2 \leq f'(x) \leq 4$ for all x . Show that $6 \leq f(2) - f(-1) \leq 12$.
 (b) Suppose that f is a differentiable function satisfying $-1 \leq f'(x) \leq 3$ for all x . Show that $-7 \leq f(10) - f(3) \leq 21$.
 (c) Show that $|\cos(b) - \cos(a)| \leq |b - a|$ for all real numbers a and b .
86. (a) Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) , that $f(a) \leq g(a)$, and that $f'(x) \leq g'(x)$ for all $x \in (a, b)$. Show that $f(b) \leq g(b)$ using the Mean Value Theorem. (Hint: Consider the function h defined by $h(x) = g(x) - f(x)$.)

2.9 How derivatives affect the shape of a graph

87. Find the local maxima and minima of the function. You must state the methods you are using (such as the first or second derivative test), and you must explain why they can be applied.
- (a) $f(x) = x^3 - 3x + 3$ (b) $f(x) = x + \sqrt[3]{x}$ (c) $f(x) = \frac{x^2}{x+1}$
88. Find the intervals where f is increasing or decreasing, and find the local maxima and minima of f .
- (a) $f(x) = 2x^3 - 3x^2 - 36x$ (b) $f(x) = 4x^3 - 9x^2 + 6x + 1$
- (c) $f(x) = x^4 - 2x^2 - 3$ (d) $f(x) = \frac{x^2}{x^2 - 2}$
- (e) $f(x) = x + 2\cos(x), x \in [0, 2\pi]$ (f) $f(x) = x - \sqrt{x}, x \in [0, \infty)$

2.10 Exponential functions and their derivatives

89. Find the domain of the function.

(a) $f(x) = \frac{1 - e^{x^2}}{1 - e^{4-x^2}}$

(b) $g(x) = \frac{1 + x}{e^{\sin x}}$

(c) $h(x) = \frac{1 + x}{1 - e^{\sin x}}$

(d) $k(t) = \sqrt{1 - e^t}$

90. Differentiate the function.

(a) $f(x) = e^{-4}$

(b) $j(s) = e^s - s^e$

(c) $f(x) = (x^4 - 2x)e^x$

(d) $y = e^{bx^4}$

(e) $y = e^{2t} \sin(4t)$

(f) $y = e^{-e^x}$

(g) $y = \frac{a + be^{-x}}{c + de^{-x}}$

(h) $y = \sqrt{1 - xe^{2x}}$

(i) $y = \cos^2(e^{\cos^2 t})$

91. Find the equation of the tangent line to the curve with equation $xe^y - ye^{-x} = 1$ through the point $(1, 0)$.

2.11 Logarithmic functions

92. Find the domain of f and give an expression for f^{-1} .

(a) $f(x) = \sqrt{1 - e^{-x}}$

(b) $f(x) = \ln(3 - \ln x)$

(c) $f(x) = \ln(e^x - 1)$

2.12 Derivatives of logarithmic functions

93. Differentiate the function.

(a) $f(x) = x + x \ln x$

(b) $f(x) = \cos(\ln x)$

(c) $f(x) = \ln(\cos x)$

(d) $f(x) = \sqrt[4]{\ln x}$

(e) $f(x) = \log_{10}(x^2 + 1)$

(f) $f(x) = x^4 + 4^x$

(g) $g(x) = \ln\left(\frac{x}{\sqrt{x^2 - 1}}\right)$

(h) $h(x) = \ln(x - \sqrt{x^2 - 1})$

(i) $y = \ln(e^x - xe^x)$

2.13 Inverse functions

94. (a) How do you show that a function is one-to-one?

(b) How do you show that a function is *not* one-to-one?

95. Determine whether the function is one-to-one.

(Unless stated otherwise, the function has the largest possible domain.)

(a) $f(x) = \frac{1}{3}(x + 4)$

(b) $f(x) = x^2 - 4x + 1$

(c) $h(x) = |x|$

(d) $g(x) = \sqrt[4]{x}$

(e) $h(x) = 1 + \sin(x)$

(f) $k(x) = 1 + \sin(x)$ with domain $[-\pi/2, \pi/2]$.

96. (a) If $f(x) = x^3 + x^2 + x$, find $f^{-1}(3)$ and $f^{-1}(-1)$.

(b) If $h(x) = x + \sqrt[3]{x}$, find $h^{-1}(10)$ and $h^{-1}(2)$.

97. Given a one-to-one function $f(x)$, how do you find a formula for the inverse function $f^{-1}(x)$?

98. Find a formula for the inverse of the function.

(Unless stated otherwise, the function has the largest possible domain.)

(a) $f(x) = 4 - 3x$

(b) $f(x) = 1 - \sqrt{2x + 1}$

(c) $g(x) = x^2 - 2x$ with domain $[1, \infty)$.

(d) $h(x) = \frac{1 - 2\sqrt{x}}{1 + 2\sqrt{x}}$

(e) $k(t) = 4t^2 - 2t$ with domain $[1/4, \infty)$.

(f) $f(x) = \frac{2x - 1}{4x + 1}$

99. Find $(f^{-1})'(a)$. (In all cases the function is one-to-one on an open interval containing a .)
- (a) $f(x) = 3x^3 + 2x^2 + 8x + 3$, $a = 3$
- (b) $f(x) = x^4 + 2 \sin x + 3 \cos x$, $a = 3$
- (c) $f(x) = 4 - x^3 + \tan(2x)$, $a = 4$
- (d) $f(x) = \sqrt{\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}}$, $a = 1$
100. Suppose that f is a one-to-one function satisfying $f(3) = 2$, $f(2) = 1$, $f'(2) = 3$ and $f'(3) = 4$. Define a function G by $G(x) = \frac{f(x)^2}{f^{-1}(x)}$. Compute $G'(2)$.

3.3 Substitution rule

101. Solve

$$\int (2t - 7)^3 dt$$

in two ways:

- (a) By first expanding $(2t - 7)^3$.
- (b) By substituting $u = 2t - 7$.

102. Solve

$$\int \sin(x)^3 dx$$

in two ways:

- (a) By using angle-sum trig identities to simplify $\sin(x)^3$, and then integrate the result.
- (b) By substituting $u = \cos(x)$. (It is helpful to recall that $\sin(x)^2 = 1 - \cos(x)^2$.)

103. Solve using integration by substitution:

- (a) $\int \sin(t) (\cos(t)^4 + 6 \cos(t)^3 - 6) dt.$
- (b) $\int \frac{1}{x} (\ln(x) - \ln(x)^3 + 2 \cos(\ln(x))) dx.$
- (c) $\int w^w (1 + \ln(w)) dw.$
- (d) $\int \frac{x \cos(3x^2 + 5)}{\sin(3x^2 + 5) + 6} dx.$
- (e) $\int \sec(x)^2 \frac{e^{\sqrt{\tan(x)+17}}}{\sqrt{\tan(x) + 17}} dx.$

104. Let $A, B, C \in \mathbb{R}$ be constants, with $B, C > 0$. Solve:

$$\begin{aligned} \text{(a)} \quad & \int \frac{A}{B + Cx} dx. \\ \text{(b)} \quad & \int \frac{A}{B + Cx^2} dx. \\ \text{(c)} \quad & \int \frac{A}{\sqrt{B - Cx^2}} dx. \end{aligned}$$

105. For the purpose of this question, we use the following. There exists a differentiable function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ which satisfies

$$\rho'(x) = e^{-x^2}$$

for all $x \in \mathbb{R}$. Find the following indefinite integrals (the answers will involve ρ).

$$\begin{aligned} \text{(a)} \quad & \int \frac{2e^{-x^2}}{\sqrt{\pi}} dx. \\ \text{(b)} \quad & \int xe^{-1+2x^2-x^4} dx. \\ \text{(c)} \quad & \int t^2 \rho(t^3) e^{-t^6} dt. \end{aligned}$$

(The function $\frac{2}{\sqrt{\pi}}\rho$ is called the *error function*, and is fundamentally important in statistics.)

3.4 Area and the definite integral

106. Find the area of the following:

- The region enclosed by $y = \sin(x)$, $x = 0$, $x = \pi$, and $y = 0$.
- The region enclosed by the curves $y = x^3 - 25x$ and $y = -x^3 - x + 32$.
- The region enclosed by the curves $x = -e^y \sin(e^y)$, and $y = -x$, $y = \ln(\pi)$, and $y = \ln(2\pi)$.
- The region enclosed by the curves $y = \frac{x^2\sqrt{10-x^3}}{3}$ and $x = \frac{y^2\sqrt{10-y^3}}{3}$, between the points $(0, 0)$ and $(1, 1)$. (These curves do not intersect between these points – you can use this fact without proving it.)

107. Differentiate the following functions:

$$(a) \quad f(x) = \int_3^{2x} e^{-t^2} dt.$$

$$(b) \quad g(x) = \int_{-x}^{x^2} \sqrt[3]{\cos(t)} dt.$$

$$(c) \quad h(x) = \int_x^0 -x^2 \sqrt{1+t^3} dt$$

The x^2 is **not** meant to be t^2 in part (c).

108. Compute the following definite integrals:

$$(a) \quad \int_0^{5/2} \frac{1}{4x^2 + 25} dx.$$

$$(b) \quad \int_0^{5/2} \frac{x}{4x^2 + 25} dx.$$

$$(c) \quad \int_1^4 \frac{x^2 - \sqrt{x}}{(x^3 - 2x^{3/2} + 2)^2} dx.$$

$$(d) \quad \int_0^{\pi/2} \sin(x) \cdot \cos(x) \cdot \cos\left(\frac{\pi}{e-1} (e^{1-\cos(x)^2} - 1)\right) \cdot e^{1-\cos(x)^2} dx.$$

3.6 Integration by parts

109. Solve the following indefinite integrals using Integration by Parts.

$$(a) \quad \int x^2 \cos(2\pi x) dx.$$

$$(b) \quad \int x\sqrt{2x+1} dx.$$

$$(c) \quad \int e^{2x} \cos(e^x) dx.$$

$$(d) \quad \int x \ln(x)^2 dx.$$

$$(e) \quad \int \ln(x)^3 dx.$$

3.7 Integrals containing trigonometric functions

110. Solve the following trigonometric integrals:

$$(a) \int \sin(t)^3 \cos(t)^3 dt.$$

$$(b) \int \sin(x) \cos(2x)^2 dx.$$

$$(c) \int \sin(x) \sin(x+2) dx.$$

$$(d) \int \tan(x) \sec(x)^2 dx.$$

$$(e) \int \tan(x)^2 \sec(x)^2 dx.$$

$$(f) \int \frac{1}{\sin(x)} dx.$$

$$(g) \int \cot(x)^2 \csc(x) dx.$$

3.8 Trigonometric substitutions

111. Solve the following integrals. These may require a combination of techniques.

$$(a) \int x \sec(x)^2 dx.$$

$$(b) \int x^3 \sqrt[3]{1+x^2} dx.$$

$$(c) \int x \cos(x) e^x dx.$$

$$(d) \int e^{\sqrt[3]{3x+7}} dx.$$

$$(e) \int \sin(x) \ln((\sin(x))^2) dx.$$

112. Prove that, for any $n \in \mathbb{N}$,

$$\int \ln(x)^n dx = n! x \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} \ln(x)^k + C.$$

3.9 Integrating rational functions: partial fraction decomposition

113. Solve the following integrals.

(a) $\int \frac{1}{(x^2 - 1)^{3/2}} dx.$

(b) $\int x\sqrt{-(x^2 + 4x + 3)} dx.$

(c) $\int \frac{1}{(3t - 1)^2\sqrt{9t^2 - 6t + 5}} dt.$

114. Find the following integrals using Partial Fraction Decomposition.

(a) $\int \frac{x^3 + x^2 - 13x - 63}{x^2 - x - 12} dx.$

(b) $\int \frac{6x^2 + 49x - 31}{x^3 + 8x^2 + 5x - 14} dx.$

(c) $\int \frac{x^3 - 2x^2 - 9x + 16}{x^3 - 6x^2 + 12x - 8} dx.$

(d) $\int \frac{x^2 - 4x + 14}{x^2 - 6x + 18} dx.$

UNIVERSITY OF ABERDEEN

ADVANCED MATHEMATICS I-1
ANSWERS AND WORKED SOLUTIONS

1. (a) -1.25 .
(b) f has domain $[-2, 4]$ and range $[-2, 3]$.
(c) g has domain $[-3, 4]$ and range $[-0.25, 4]$. (Here -0.25 is an estimate.)
2. (a) $\{x \mid x \neq -2, 2\}$, or in other words $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
(b) \mathbb{R} .
(c) $\{x \mid x \neq 1, -2\}$, or in other words $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
(d) *Worked Solution:* $\sqrt{2-t}$ is defined when $2-t \geq 0$, or in other words when $t \leq 2$. $\sqrt{2+t}$ is defined when $2+t \geq 0$, or in other words when $t \geq -2$. So $\sqrt{2-t} + \sqrt{2+t}$ is defined when both condition $st \geq -2$ and $t \leq 2$ are true. Thus the range is $[-2, 2]$.
(e) $(-\infty, 0) \cup (6, \infty)$.

(f) *Worked Solution:* $\frac{1}{\sqrt[3]{u^2-6u}}$ is defined when $\sqrt[3]{u^2-6u}$ is defined and nonzero. Now $\sqrt[3]{u^2-6u}$ is always defined, and it is nonzero when u^2-6u is nonzero, or in other words when $u \neq 0, 6$. Thus the domain is $\{u \mid u \neq 0, 6\}$, or in other words $(-\infty, 0) \cup (0, 6) \cup (6, \infty)$.

(g) *Worked Solution:* For the function to be defined we require that both square roots exist, or in other words that

$$1+t \geq 0$$

for $\sqrt{1+t}$ to exist, and that

$$4 - \sqrt{1+t} \geq 0$$

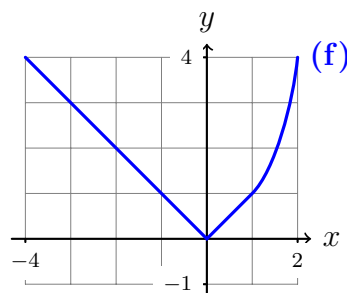
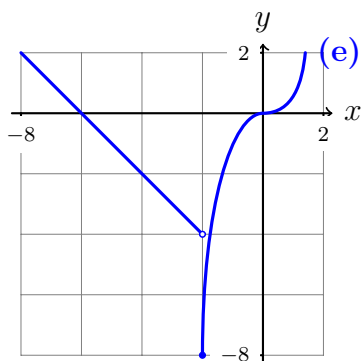
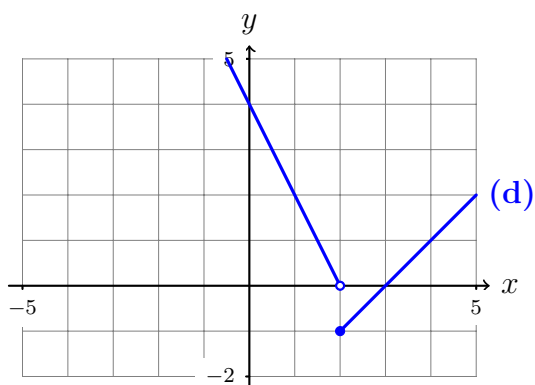
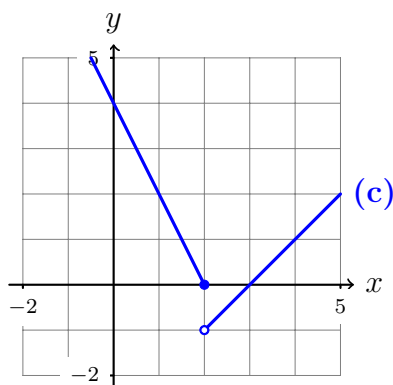
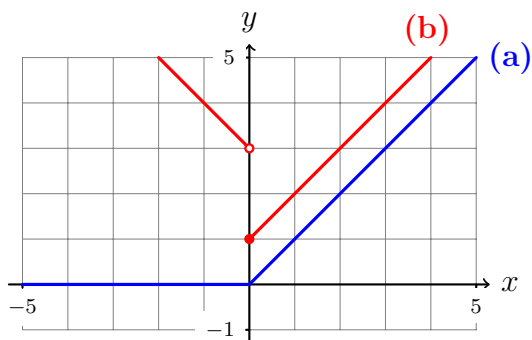
for $\sqrt{4 - \sqrt{1+t}}$ to exist. The first of these states that $t \geq -1$, and the second is equivalent to $4 \geq \sqrt{1+t}$, which is equivalent to $16 \geq 1+t$, which is equivalent to $t \leq 15$. So altogether the requirement is that

$$-1 \leq t \leq 15.$$

So the domain is $[-1, 15]$.

(h) The domain is $[-1, 0) \cup (0, 15]$.

3.



4. (a) $f(x) = \begin{cases} 1-x, & 0 \leq x \leq 1 \\ x-1, & 1 < x \leq 3 \end{cases}$ and the range is $[0, 2]$.
- (b) $f(x) = \begin{cases} x+2, & -2 \leq x < -1 \\ -x, & -1 \leq x < 1 \\ x-2, & 1 \leq x \leq 2 \end{cases}$ and the range is $[-1, 1]$.

(c) *Worked Solution:* In the range $-2 \leq x \leq 0$ the graph is the line with gradient $3/2$ and y -intercept 3 , so the function is given by $\frac{3}{2}x + 3$ in this range. And in the range $0 \leq x \leq 2$ the graph is the line with gradient $-3/2$ and y -intercept 3 , so the function is given by $3 - \frac{3}{2}x$ in this range. So we have the following:

$$f(x) = \begin{cases} \frac{3}{2}x + 3, & -2 \leq x < 0 \\ 3 - \frac{3}{2}x, & 0 \leq x \leq 2 \end{cases}$$

We have chosen $x < 0$ in the first line to avoid defining $f(x)$ twice. But we could just

as well have written the following, and the answer would still be right.

$$f(x) = \begin{cases} \frac{3}{2}x + 3, & -2 \leq x < 0 \\ 3 - \frac{3}{2}x, & 0 \leq x \leq 2 \end{cases}$$

By looking at which y -values are attained by the function we see that the range is $[0, 3]$.

(d) $f(x) = \begin{cases} -2, & -2 \leq x < 0 \\ 2, & 0 \leq x < 2 \end{cases}$ and the range is $[-2, 2]$.

5. (a) The function is odd. It is not even.
 (b) The function is neither odd nor even.
 (c) The function is even. It is not odd.
 (d) The function is even. It is not odd.
 (e) The function is both odd and even.
6. (a) The function is even. It is not odd.
 (b) The function is odd. It is not even.
 (c) The function is neither even nor odd.

(d) *Worked Solution:* $f(x) = \frac{x^2}{x^3 + x}$ so that

$$f(-x) = \frac{(-x)^2}{(-x)^3 - x} = \frac{x^2}{-x^3 - x} = -\frac{x^2}{x^3 + x} = -f(x)$$

and consequently f is odd. Since $f(-x) \neq f(x)$, it is not even.

- (e) The function is even. It is not odd.
 (f) The function is neither even nor odd.
 (g) The function is odd. It is not even.
 (h) *Worked Solution:* $g(x) = |x| \cdot x^2$ so that

$$g(-x) = |-x| \cdot (-x)^2.$$

Now $|-x| = |x|$ and $(-x)^2 = (-x)(-x) = x^2$, so that

$$g(-x) = |-x| \cdot (-x)^2 = |x| \cdot x^2 = g(x).$$

Thus g is even. Since $g(x) \neq -g(x)$, it is not odd.

- (i) The function is odd. It is not even.
 (j) *Worked Solution:* We work out $(p+q)(-x)$ and see whether it is equal to $(p+q)(x)$ or $-(p+q)(x)$. We find that

$$(p+q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p+q)(x)$$

so that $p + q$ is even. It is not (necessarily) odd.

(k) The function is odd. It is not (necessarily) even.

(l) The function is odd.

(m) *Worked Solution:* Since q is even and r is odd we have

$$(q \circ r)(-x) = q(r(-x)) = q(-r(x)) = q(r(x))$$

so that $q \circ r$ is even.

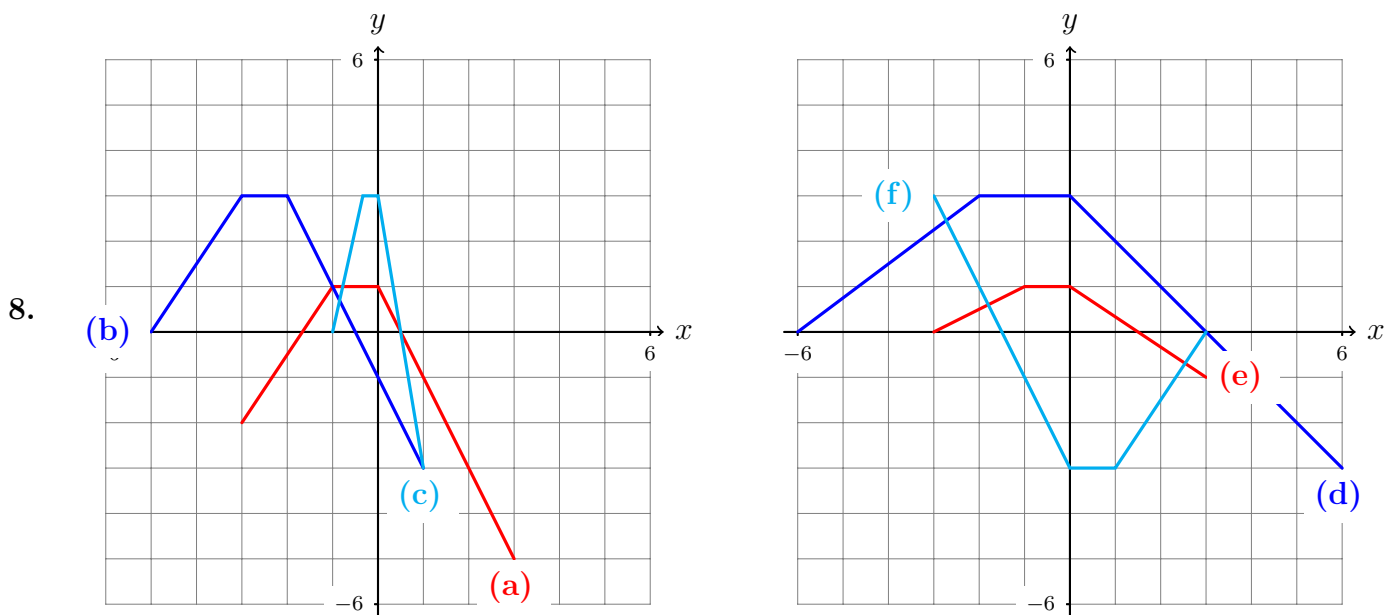
(n) The function is even.

(o) The function is odd.

7. *Worked Solution for g:* Recall that $g(x) = \frac{f(x)+f(-x)}{2}$, so that

$$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = g(x),$$

and consequently g is even.



9. (a) $g(x) = f(x - 2) - 1$.

(b) *Worked Solution:* The graph of $h(x)$ is obtained from the graph of f by the following steps: First, reflect the graph of f in the x -axis. Second, move it 2 to the right. Third, move it 1 down. After the first step, we have the graph of the function $-f(x)$. After the second step, we have the graph of the function $-f(x - 2)$. After the third step, we have the graph of the function $-f(x - 2) - 1$. So $h(x) = -f(x - 2) - 1$.

(c) $k(x) = -\frac{1}{3}f(-x - 2)$.

10. (a) $(f + g)(x) = x^3 + x^2 - 2$, and its domain is \mathbb{R} .
 (b) $(f - g)(x) = x^3 - x^2 + 6$, and its domain is \mathbb{R} .
 (c) $(fg)(x) = (x^3 + 2)(x^2 - 4)$, and its domain is \mathbb{R} .
 (d) *Worked Solution:*

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x^3 + 2}{x^2 - 4}$$

Its domain is the collection of all points x that are in the domain of f , and the domain of g , and that satisfy $g(x) \neq 0$. The domain of f and g are both \mathbb{R} . And $g(x) = 0$ if and only if $x = 2$ or $x = -2$. So the domain of f/g is $\{x \mid x \neq 2, -2\}$ or in other words $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

11. (a) *Worked Solution:* $(f + g)(x) = f(x) + g(x) = \sqrt{5 - x} + \sqrt{x^2 - 4}$. The domain of $f + g$ is the set of all x that lie in the domain of f and the domain of g . The domain of f is the set of points x such that $5 - x \geq 0$, or in other words $(-\infty, 5]$. And the domain of g is the set of points x such that $x^2 - 4 \geq 0$, or in other words $(-\infty, -2] \cup [2, \infty)$. The intersection of the two domains is then the intersection of $(-\infty, 5]$ with $(-\infty, -2] \cup [2, \infty)$, and this is exactly $(-\infty, -2] \cup [2, 5]$.
 (b) $(f - g)(x) = \sqrt{5 - x} - \sqrt{x^2 - 4}$. The domain is $(-\infty, -2] \cup [2, 5]$.
 (c) $(fg)(x) = \sqrt{5 - x}\sqrt{x^2 - 4}$. The domain is $(-\infty, -2] \cup [2, 5]$.
 (d) $(f/g)(x) = \frac{\sqrt{5-x}}{\sqrt{x^2-4}}$. The domain is $(-\infty, -2] \cup (2, 5]$.

12. In every part, the domain is \mathbb{R} .

(a) $(f \circ g)(x) = 3x^2 - 1$

(b) *Worked Solution:* $(g \circ f)(x) = g(f(x)) = g(3x + 2) = (3x + 2)^2 - 1 = 9x^2 + 12x + 3$. The domain consists of all x such that $f(x)$ is defined and $g(f(x))$ is defined. Since both g and f are defined for all x , it follows that $g \circ f$ is too, hence its domain is \mathbb{R} .

(c) $(g \circ g)(x) = x^4 - 2x^2$.

(d) $(f \circ f)(x) = 9x + 8$.

13. In every part, the domain is \mathbb{R} .

(a) $(f \circ g)(x) = \sin(2x + 2)$.

(b) $(g \circ f)(x) = 2 \sin x + 2$.

(c) $(g \circ g)(x) = g(g(x)) = g(2x + 2) = 2(2x + 2) + 2 = 4x + 6$. Its domain consists of all x for which $g(x)$ and $g(g(x))$ are defined. Since $g(x)$ is defined for all x , it follows that $g \circ g$ is as well. Hence its domain is \mathbb{R} .

(d) $(f \circ f)(x) = \sin(\sin(x))$.

14. (a) $(f \circ g)(x) = x + 2$, and its domain is $\{x \mid x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$.

(b) *Worked Solution:* $(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)+2} = \frac{1}{\frac{1}{x}+2} = \frac{x}{2x+1}$. Its domain consists of all x for which $f(x)$ and $g(f(x))$ are defined. In other words, it consists of all x for

which $x \neq 0$ and $1/x \neq -2$. And that means that it consists of all x for which $x \neq 0$ and $x \neq -1/2$. So the domain is $\{x \mid x \neq 0, x \neq -1/2\} = (-\infty, -1/2) \cup (-1/2, 0) \cup (0, \infty)$.

(c) $(g \circ g)(x) = \frac{x+2}{2x+5}$. Its domain is $\{x \mid x \neq -2, x \neq -5/2\}$.

(d) $(f \circ f)(x) = x$, and its domain is $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

15. (a) $f(8)$ does not exist because f is not defined at 8.
 (b) $\lim_{x \rightarrow 4^-} f(x)$ does not exist because we may find $x < a$ arbitrarily close to a such that $f(x) = 6$, and we may find $x < a$ arbitrarily close to a such that $f(x) = 2$.
 (c) $\lim_{x \rightarrow 4^+} f(x) = 4$.
 (d) $\lim_{x \rightarrow 4} f(x)$ does not exist for the same reasons that $\lim_{x \rightarrow 4^-} f(x)$ does not exist.
 (e) $\lim_{x \rightarrow 0^+} f(x) = 2$.
 (f) $\lim_{x \rightarrow 0^-} f(x) = 6$.
 (g) $\lim_{x \rightarrow 0} f(x)$ does not exist because $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.
 (h) $f(0) = 2$.
 (i) $\lim_{x \rightarrow -4^-} f(x) = 2$.
 (j) $\lim_{x \rightarrow -4^+} f(x) = 4$.
 (k) $f(-4) = 3$.
 (l) $\lim_{x \rightarrow 8} f(x) = 6$.

16. (a) $\lim_{x \rightarrow -4^-} f(x) = \infty$.
 (b) $\lim_{x \rightarrow -4^+} f(x) = -\infty$.
 (c) $\lim_{x \rightarrow -4} f(x)$ does not exist.
 (d) $\lim_{x \rightarrow 4^-} f(x) = -\infty$.
 (e) $\lim_{x \rightarrow 4^+} f(x) = -\infty$.
 (f) $\lim_{x \rightarrow 4} f(x) = -\infty$.

17. (a) *Worked Solution:* We use the fact that $\lim_{x \rightarrow -1} h(x)$ exists if and only if $\lim_{x \rightarrow -1^-} h(x)$ and $\lim_{x \rightarrow -1^+} h(x)$ both exist and are equal, in which case

$$\lim_{x \rightarrow -1} h(x) = \lim_{x \rightarrow -1^-} h(x) = \lim_{x \rightarrow -1^+} h(x).$$

Since $h(x) = 2 + x$ for $x < -1$ we have

$$\lim_{x \rightarrow -1^-} h(x) = \lim_{x \rightarrow -1^-} 2 + x = 2 + (-1) = 1$$

and since $h(x) = x^2$ for $-1 < x < 1$ we have

$$\lim_{x \rightarrow -1^+} h(x) = \lim_{x \rightarrow -1^+} x^2 = (-1)^2 = 1.$$

Since these limits both exist and are equal, we have

$$\lim_{x \rightarrow -1} h(x) = 1.$$

(b) The limit does not exist.

(c) *Worked Solution:*

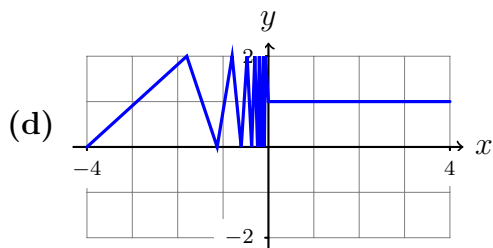
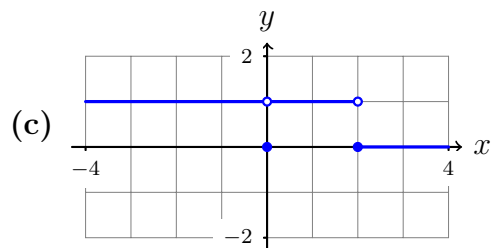
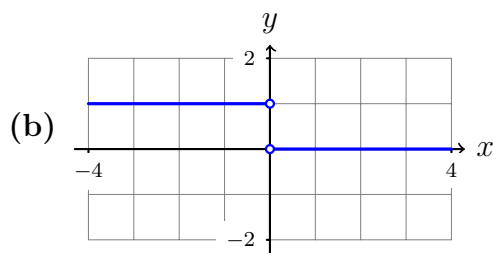
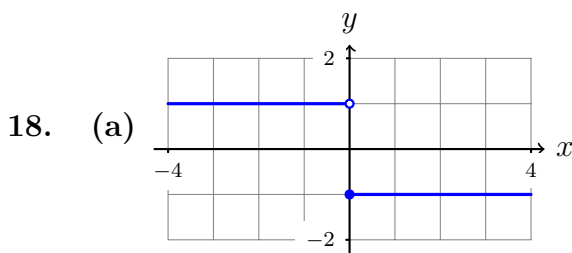
$\lim_{x \rightarrow a} f(x)$ exists for all a except possibly $x = -1$ and $x = 1$.

Now $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (-2 - x) = -1$ and $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (x^3) = -1$. Since $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$, it follows that $\lim_{x \rightarrow -1} f(x)$ exists.

Next, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^3 = 1$, while $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 1 - x = 0$. Since $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$, it follows that $\lim_{x \rightarrow 1} f(x)$ does not exist.

So $\lim_{x \rightarrow a} f(x)$ exists for all a except $a = 1$.

(d) The limit $\lim_{x \rightarrow a} f(x)$ exists for all x except $a = 0$.



19. (a) $\lim_{x \rightarrow \pi/2^-} \tan x = \infty$ (b) $\lim_{x \rightarrow \pi/2^+} \tan x = -\infty$ (c) $\lim_{x \rightarrow 3^+} \frac{1+x}{x-3} = \infty$

(d) $\lim_{x \rightarrow 3^-} \frac{1+x}{x-3} = -\infty$ (e) $\lim_{x \rightarrow 0} \frac{1+x}{x^2(x-3)} = -\infty$

(f) *Worked Solution:*

As x approaches 3 the terms $1 + x$ and x^2 approach 4 and 9 respectively, while $x - 3$ approaches 0. Thus $\frac{1+x}{x^2(x-3)}$ grows without bound as x approaches 3.

However, if $x > 3$ then the term $1 + x$ is positive, as are x^2 and $x - 3$. Thus $\frac{1+x}{x^2(x-3)}$ is always positive for $x > 3$.

It follows that $\lim_{x \rightarrow 3^+} \frac{1+x}{x^2(x-3)} = \infty$.

20. (a) 6

(b) 16

(c) $-\sqrt[3]{3}$

(d) *Worked Solution:* First, $\lim_{x \rightarrow 1} 4h(x) = 4 \lim_{x \rightarrow 1} h(x) = 4 \times -3 = -12$, by the law for functions multiplied by a scalar. Since this limit exists and is nonzero, and since $\lim_{x \rightarrow 1} f(x)$ also exists, we may use the limit law for quotients to see that $\lim_{x \rightarrow 1} \frac{f(x)}{4h(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} 4h(x)} = \frac{2}{-12} = -1/6$.

(e) 0

21. (a) 31

(b) 15

(c) *Worked Solution:* For $u = 3$ we have $u^3 - 3u + 3 = 21$. So for u close to 3 we have $u^3 - 3u + 3 > 0$, and so we may use the n -th root law to see that

$$\lim_{u \rightarrow 3} \sqrt{u^3 - 3u + 3} = \sqrt{\lim_{u \rightarrow 3} (u^3 - 3u + 3)}$$

and by the direct substitution law this is equal to $\sqrt{3^3 - 3u + 3} = \sqrt{21}$.

(d) $\sqrt{\frac{3}{5}}$

22. The left hand side of the first equation is only defined for $x \neq 3$, while the right hand side is defined for all x . This is nevertheless enough to show that the two limits are equal.

23. (a) 1

(b) Does not exist.

(c) 7/4

(d) Does not exist.

(e) -4

(f) 12

(g) Does not exist.

(h) $1/\sqrt{2}$

(i) *Worked Solution:* First we simplify the function for $t \neq 0$:

$$\frac{1}{t} - \frac{2}{t^2 + 2t} = \frac{t + 2}{t^2 + 2t} - \frac{2}{t^2 + 2t} = \frac{t + 2 - 2}{t^2 + 2t} = \frac{1}{t + 2}$$

Thus

$$\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{2}{t^2 + 2t} \right) = \lim_{t \rightarrow 0} \left(\frac{1}{t + 2} \right) = \frac{1}{2}.$$

(j) 1/54

24. *Worked Solution:*

(a) We will show that the left and right handed limits both exist and are zero. Indeed, since $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$, we have

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

and

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

so that $\lim_{x \rightarrow 0} |x|$ exists and is equal to 0.

(b) We do this in three parts, depending on the value of x .

First suppose that $x = 0$. Then $-|x| = 0$, $f(x) = 0$ and $|x| = 0$. So then $-|x| \leq f(x) \leq |x|$ certainly holds.

Next suppose that $x > 0$. In this case $|x| = x$. Then since $-1 \leq \cos(1/x) \leq 1$ and $x > 0$, we have $-x \leq x \cos(1/x) \leq x$. In other words, $-|x| \leq f(x) \leq |x|$.

Finally suppose that $x < 0$. In this case $|x| = -x$. Since $-1 \leq \cos(1/x) \leq 1$ and since $x < 0$, multiplying the inequality by x reverses the inequalities, so we have $-x \geq x \cos(1/x) \geq x$, or in other words $x \leq x \cos(1/x) \leq -x$, or in other words $-|x| \leq f(x) \leq |x|$, as required.

(c) We know that $\lim_{x \rightarrow 0} (-|x|) = 0$, $\lim_{x \rightarrow 0} |x| = 0$, and $-|x| \leq f(x) \leq |x|$ for all x . So the squeeze theorem applies and shows us that $\lim_{x \rightarrow 0} f(x) = 0$ as required.

25. (a) *Worked Solution:* Let us define functions p and q by $p(x) = x^2 - 5x + 8$ and $q(x) = 2x^2 - 11x + 17$. Then the question tells us that $p(x) \leq f(x) \leq q(x)$. Since p and q are polynomials, direct substitution shows that $\lim_{x \rightarrow 3} p(x) = p(3) = 2$ and $\lim_{x \rightarrow 3} q(x) = q(3) = 2$. So the squeeze theorem tells us that $\lim_{x \rightarrow 3} f(x) = 2$.

(b) 1

26. (a) Does not exist.

(b) 3

(c) *Worked Solution:* For $x < 0$ we have $\frac{|x|-3}{x+3} = \frac{-x-3}{x+3} = -1$ so that $\lim_{x \rightarrow -3} \left(\frac{|x|-3}{x+3} \right) = \lim_{x \rightarrow -3} -1 = -1$.

28. *Worked Solution:* We have $\lim_{x \rightarrow 0} \left(\frac{g(x)}{x^2} \right) = \lim_{x \rightarrow 0} \left(x \cdot \frac{g(x)}{x^3} \right) = \lim_{x \rightarrow 0} (x) \cdot \lim_{x \rightarrow 0} \left(\frac{g(x)}{x^3} \right) = 0 \cdot 4 = 0$. Similarly, $\lim_{x \rightarrow 0} \frac{g(x)}{x} = 0$.

29. $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$ serve as answers to both parts.

- 30.** The example asks you to show that $\lim_{x \rightarrow a} f(x) = L$ where $a = 2$, $f(x) = 2x + 3$ and $L = 7$. So given $\epsilon > 0$ we must specify a $\delta > 0$ and show that if $|x - 2| < \delta$ then $|(2x + 3) - 7| < \epsilon$.

So the error is that the solution proves $|(2x + 3) - 7| < 2\epsilon$. To make a correct solution we can choose a different value of δ and follow through the same reasoning. In this case $\delta = \epsilon/2$ works. (If you can't see why this works, then try defining $\delta = c \cdot \epsilon$ and working through the steps of the solution, then check that by choosing $c = 1/2$ the computation ends as " $< \epsilon$ ".

- 31.** The question asks us to show that $\lim_{x \rightarrow a} f(x) = L$, where $a = 3$, $f(x) = 3 - 5x$, and $L = -12$. So given $\epsilon > 0$, we must define a $\delta > 0$ and show that when $|x - a| < \delta$ we have $|f(x) - L| < \epsilon$.

The solution seems to do this. However there are two wrong steps: first, $|(-5)(3 - x)| = |-5| \times |3 - x| = 5|3 - x|$, so there is a sign error in the third =, and $|3 - x| = |x - 3|$, so there is another sign error in the fourth =. Nothing else needs to be changed.

- 32.** (c) *Pre-Solution:* Given $\epsilon > 0$ we must find $\delta > 0$ such that:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \left| \left(\frac{1}{2}x - 3 \right) - (-2) \right| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \left| \frac{1}{2}x - 1 \right| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } \frac{1}{2}|x - 2| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x - 2| < \delta \text{ then } |x - 2| < 2\epsilon$$

If we choose $\delta = 2\epsilon$ then this will certainly be true.

Solution: Given $\epsilon > 0$, let $\delta = 2\epsilon$. Then if $0 < |x - 2| < \delta$, we have

$$\left| \left(\frac{1}{2}x - 3 \right) - (-2) \right| = \left| \frac{1}{2}x - 1 \right| = \frac{1}{2}|x - 2| < \frac{1}{2}\delta = \frac{1}{2}2\epsilon = \epsilon.$$

Thus $\lim_{x \rightarrow 2} \left(\frac{1}{2}x - 3 \right) = -2$ as claimed.

- 33.** (a) *Solution:*

Given $\epsilon > 0$, let $\delta = \epsilon$. Then if $0 < |x - a| < \delta$, we have $|x - a| < \delta = \epsilon$. So $\lim_{x \rightarrow a} x = a$.

(f) *Pre-Solution:* Given $\epsilon > 0$, we must find $\delta > 0$ such that:

$$\text{if } 0 < |x - (-5)| < \delta, \text{ then } |x^2 - 25| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x + 5| < \delta, \text{ then } |(x - 5)(x + 5)| < \epsilon$$

Or in other words:

$$\text{if } 0 < |x + 5| < \delta, \text{ then } |x - 5| \cdot |x + 5| < \epsilon$$

So we have to choose δ so that if $|x + 5| < \delta$ then $|x - 5|$ and $|x + 5|$ are small enough. Let's suppose that $\delta \leq 1$. Then if $0 < |x + 5| < \delta$, we have $0 < |x + 5| < 1$, so that $-1 < x + 5 < 1$, and consequently $-11 < x - 5 < -9$, and consequently $|x - 5| < 11$.

So we know that if $\delta \leq 1$ and $0 < |x - (-5)| < \delta$, then $|x^2 - 25| = |x - 5| \cdot |x + 5| < 11\delta$. Then if $\delta \leq \epsilon/11$, that will be enough. We can arrange this by taking $\delta = \min(1, \epsilon/11)$.

Solution: Given $\epsilon > 0$, let $\delta = \min(1, \epsilon/11)$. Suppose that $0 < |x - (-5)| < \delta$, or in other words that $0 < |x + 5| < \delta$. Then $|x + 5| < 1$, so that $-1 < x + 5 < 1$, and consequently $-11 < x - 5 < -9$, and consequently $|x - 5| < 11$. So

$$|x^2 - 25| = |(x - 5)(x + 5)| = |x - 5| \cdot |x + 5| < 11 \cdot \delta \leq 11 \cdot \epsilon/11 = \epsilon.$$

Or in other words

$$|x^2 - 25| < \epsilon.$$

We have shown that $\lim_{x \rightarrow -5} x^2 = 25$ as required.

- 34.** (a) If $a \leq b$ then $\min(a, b) = a$. Since $a \leq a$ and $a \leq b$, we have $\min(a, b) \leq a$ and $\min(a, b) \leq b$ as required. If $b < a$ then a similar argument shows that the same inequalities hold.

The inequalities are used to show that $\delta \leq 2$ in the second paragraph, and to show that $\delta \leq 3\epsilon$ in the long series of inequalities.

(b) The identity is used to show that $|2 - x| = |-(x - 2)| = |x - 2|$.

(c) If $|x - 2| < 1$ then $-1 < x - 2 < 1$, and adding 3 to all terms gives $2 < x + 1 < 4$, so that $|x + 1| > 2$ and consequently $\frac{1}{|x+1|} < \frac{1}{2}$.

If $|x - 2| < 3$ then we find that $|x + 1| > 0$, but then we can conclude nothing about $\frac{1}{|x+1|}$.

(d) Given $\epsilon > 0$, define $\delta = \min(1, 6\epsilon)$. Suppose that $0 < |x - 2| < \delta$.

Since $|x - 2| < \delta$ and $\delta \leq 1$, we have $|x - 2| < 1$. It follows that $-1 < x - 2 < 1$. By adding 3 to all terms we find that $2 < x + 1 < 4$. Consequently $|x + 1| > 2$, and rearranging gives $\frac{1}{|x+1|} < \frac{1}{2}$.

Now

$$\begin{aligned}
 \left| \frac{1}{x+1} - \frac{1}{3} \right| &= \left| \frac{3 - (x+1)}{3(x+1)} \right| \\
 &= \left| \frac{2-x}{3(x+1)} \right| \\
 &= \frac{1}{3} \cdot \frac{1}{|x+1|} \cdot |2-x| \\
 &= \frac{1}{3} \cdot \frac{1}{|x+1|} \cdot |x-2| \\
 &< \frac{1}{3} \cdot \frac{1}{2} \cdot \delta \\
 &\leq \frac{1}{3} \cdot \frac{1}{2} \cdot 6\epsilon \\
 &= \epsilon
 \end{aligned}$$

and so $\left| \frac{1}{x+1} - \frac{1}{3} \right| < \epsilon$. Thus $\lim_{x \rightarrow 2} \left(\frac{1}{x+1} \right) = \frac{1}{3}$ as required.

(e) In this case you cannot prove an inequality of the form $\frac{1}{|x+1|} < ?$ and so the solution cannot be made to work in this case.

35. (a) Solution:

Let $\epsilon > 0$. Define $\delta = \min(1, \epsilon/4)$ and let x be such that $0 < |x-2| < \delta$.

Since $\delta = \min(1, \epsilon/4)$ it follows that $\delta \leq 1$ so that $|x-2| < 1$. Thus $-1 < x-2 < 1$, so that $2 < x+1 < 4$, so that $|x+1| < 4$.

Now

$$\begin{aligned}
 |(x^2 - x - 3) - (-1)| &= |x^2 - x - 2| \\
 &= |(x+1)(x-2)| \\
 &= |x+1| \cdot |x-2| \\
 &< 4 \cdot \delta \\
 &\leq 4 \cdot \epsilon/4 \\
 &= \epsilon
 \end{aligned}$$

so that overall we have $|(x^2 - x - 3) - (-1)| < \epsilon$ as required.

(d) Solution:

Let $\epsilon > 0$. Define $\delta = \min(1, \frac{12\epsilon}{5})$ and let x be such that $0 < |x-1| < \delta$.

Since $\delta = \min(1, \frac{12\epsilon}{5})$ it follows that $\delta \leq 1$, so that $|x-1| < 1$, so that $-1 < x-1 < 1$, so that $3 < x+3 < 5$, so that $|x+3| > 3$ and consequently $\frac{1}{|x+3|} < \frac{1}{3}$.

Now

$$\begin{aligned}
 \left| \frac{x-2}{x+3} - \left(-\frac{1}{4}\right) \right| &= \left| \frac{x-2}{x+3} + \frac{1}{4} \right| \\
 &= \left| \frac{4(x-2) + (x+3)}{4(x+3)} \right| \\
 &= \left| \frac{5x-5}{4(x+3)} \right| \\
 &= \frac{5|x-1|}{4|x+3|} \\
 &< \frac{5}{4}|x-1| \frac{1}{3} \\
 &< \frac{5}{4} \cdot \frac{12\epsilon}{5} \cdot \frac{1}{3} \\
 &= \epsilon
 \end{aligned}$$

so that overall $\left| \frac{x-2}{x+3} - \left(-\frac{1}{4}\right) \right| < \epsilon$ as required.

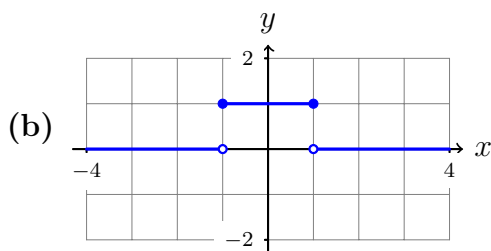
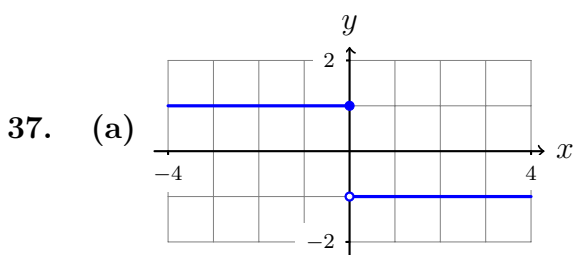
36. *Worked Solution:*

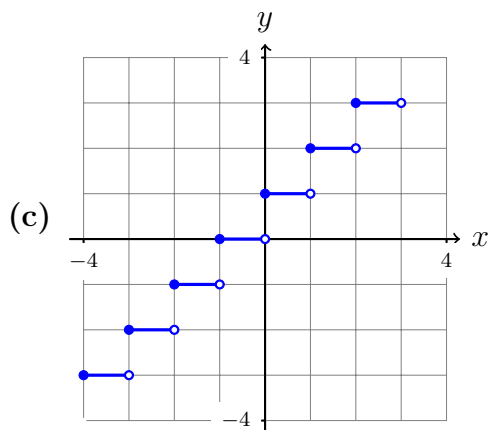
(a) -4 is a discontinuity of f because $f(-4)$ is not defined.

(b) 4 is a discontinuity of f because $\lim_{x \rightarrow 4} f(x)$ does not exist, which is because $\lim_{x \rightarrow 4^-} f(x)$ does not exist.

(c) 8 is a discontinuity of f because $\lim_{x \rightarrow 8} f(x) = 6$ but $f(8) = 2$.

(d) f is defined at all numbers x in the range $-10 \leq x \leq 10$, except for $x = -4$. It is continuous at all of those points except for $x = 4$ and $x = 8$. So it is continuous on the intervals $[-10, -4)$, $(-4, 4)$, $(4, 8)$ and $(8, 10]$.





38. (a) Because $\lim_{x \rightarrow 2} f(x) = 0$ but $f(2) = 1$.

(b) Because $\lim_{x \rightarrow 0} f(x)$ does not exist.

(c) *Worked Solution:* For f to be continuous at $a = 0$ we require that:

1. $f(0)$ is defined.
2. $\lim_{x \rightarrow 0} f(x)$ exists.
3. $\lim_{x \rightarrow 0} f(x) = f(0)$.

The first condition holds because $f(0)$ is defined to be 1. And the second condition holds because

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sin x = \sin 0 = 0$$

and

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0^2 = 0$$

are equal. However, the third condition does not hold because $\lim_{x \rightarrow 0} f(x) = 0 \neq 1 = f(0)$. Since the third condition doesn't hold, it follows that f is not continuous at 0.

39. (a) Discontinuous at 0, and continuous at neither right nor left there.

(b) No discontinuities.

(c) Discontinuous at 0, and continuous from the right but not the left there. Discontinuous at 1, and continuous from the left but not from the right there.

(d) Discontinuous at 0. Continuous at 0 from the right, but not the left. (It is continuous at 1.)

40. (a) *Worked Solution:* For f to be continuous at 1 we require that $f(1)$ is defined, that $\lim_{x \rightarrow 1} f(x)$ exists, and that $\lim_{x \rightarrow 1} f(x) = f(1)$.

- Certainly $f(1) = c \cdot 1^2 - 1 = c - 1$ is defined.

- Now we check when $\lim_{x \rightarrow 1} f(x)$ exists. We have $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 + 1 = 2$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} cx^2 - x = c - 1$. So for the limit to exist we need $2 = c - 1$, or in other words that $c = 3$.
- Finally, when the limit does exist, i.e. when $c = 3$, we have that $\lim_{x \rightarrow 1} f(x) = 2 = f(1)$.

So the final answer is that f is continuous at 1 when $c = 3$.

(b) The function is clearly continuous at every real number except perhaps -1 . In order for f to be continuous at -1 , we need that $f(-1)$ is defined, that $\lim_{x \rightarrow -1} f(x)$ exists, and that $\lim_{x \rightarrow -1} f(x) = f(-1)$. Certainly $f(-1)$ is defined; the limit exists if and only if

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$$

or in other words

$$2(-1)^2 + b(-1) = -2 + b$$

or in other words

$$b = 2;$$

and in this case the third condition holds.

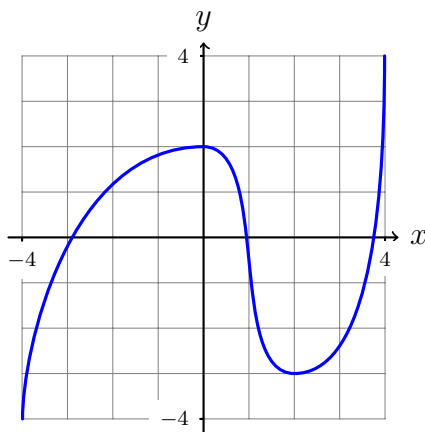
41. (b) *Worked Solution:*

Define f by $f(x) = \sin(x) - 2 \cos(x)$. It is continuous on $(-\infty, \infty)$, so it is continuous on $[-\pi/2, \pi/2]$. Also

$$\begin{aligned} f(-\pi/2) &= \sin(-\pi/2) - 2 \cos(-\pi/2) = -1 - 2 \cdot 0 = -1, \\ f(\pi/2) &= \sin(\pi/2) - 2 \cos(\pi/2) = 1 - 2 \cdot 0 = 1. \end{aligned}$$

So $f(-\pi/2) < N < f(\pi/2)$ where $N = 0$. The intermediate value theorem tells us that there is $c \in (-\pi/2, \pi/2)$ such that $f(c) = N$, or in other words such that $\sin(c) - 2 \cos(c) = 0$.

42.



43. (a) $f'(a) = 0$. (b) $f'(a) = 1$. (c) $f'(a) = 2a$.
 (d) *Worked Solution:*

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1/(a+h) - 1/a}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a - (a+h))/a(a+h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h/a(a+h)}{h} \\ &= \lim_{h \rightarrow 0} -1/a(a+h) \\ &= -1/a^2 \end{aligned}$$

44. (a) *Worked Solution:* We calculate

$$\begin{aligned} F'(2) &= \lim_{h \rightarrow 0} \frac{\frac{3}{(2+h)^2+2+h} - \frac{3}{2^2+2}}{h} = \\ \lim_{h \rightarrow 0} \frac{\frac{3}{h^2+5h+6} - \frac{1}{2}}{h} &= \lim_{h \rightarrow 0} \frac{6 - h^2 - 5h - 6}{2h(h^2 + 5h + 6)} = \\ \lim_{h \rightarrow 0} \frac{-h - 5}{2(h^2 + 5h + 6)} &= \frac{-5}{12}. \end{aligned}$$

(b) *Worked Solution:* The tangent line to the curve $y = F(x)$ at the point $(2, 1/2)$ has equation $y - 1/2 = F'(2)(x - 2)$, which simplifies to give $y = -\frac{5x}{12} + \frac{4}{3}$.

45. (a) $f'(a) = 4a - 3$

(b) *Worked Solution:*

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(a+h)^3 - 2] - [a^3 - 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2 - a^3 + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} 3a^2 + 3ah + h^2 \\ &= 3a^2 \end{aligned}$$

(c) $f'(a) = -\frac{1}{(2a+3)^2}$

$$(d) f'(a) = \frac{-1}{2\sqrt{2-a}}$$

(e) *Worked Solution:*

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sqrt{a+h+1}} - \frac{1}{\sqrt{a+1}} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sqrt{a+1} - \sqrt{a+h+1}}{\sqrt{a+h+1}\sqrt{a+1}} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(\sqrt{a+1} - \sqrt{a+h+1})(\sqrt{a+1} + \sqrt{a+h+1})}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{a+h+1})} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(a+1) - (a+h+1)}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{a+h+1})} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{a+h+1})} \right] \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{a+h+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{a+h+1})} \\ &= \frac{-1}{\sqrt{a+0+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{a+0+1})} \\ &= \frac{-1}{2\sqrt{a+1}^3} \end{aligned}$$

46. *Worked Solution:*

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} = \lim_{h \rightarrow 0} \frac{f(a+h) + c - f(a) - c}{h} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

47. This question can be answered quickly using the solutions to an earlier question about the same functions.

(a) $f'(x) = 0$. The domain of f and of f' are both \mathbb{R} .

(b) $f'(x) = 1$. The domain of f and of f' are both \mathbb{R} .

(c) $f'(x) = 2x$. The domain of f and of f' are both \mathbb{R} .

(d) $f'(x) = \frac{-1}{x^2}$. The domain of f and of f' are both $\{x \mid x \neq 0\}$.

48. (a) $f'(x) = 1 - \frac{1}{2\sqrt{x}}$. The domain of f is $[0, \infty)$ and the domain of f' is $(0, \infty)$.

(b) *Worked Solution:* The domain of g is $(-\infty, 0]$. The derivative of g is:

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{-x-h} - \sqrt{-x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(-x-h) - (-x)}{h(\sqrt{-x-h} + \sqrt{-x})} \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{-x-h} + \sqrt{-x}} \\ &= \frac{-1}{2\sqrt{-x}} \end{aligned}$$

It exists for $x \leq 0$ and $x \neq 0$, so the domain of g' is $(-\infty, 0)$.

(c) The derivative f' is given by $f'(x) = \frac{-12}{(2+3x)^2}$. The domain of f and f' are both equal to $\{x \mid x \neq \frac{-2}{3}\}$.

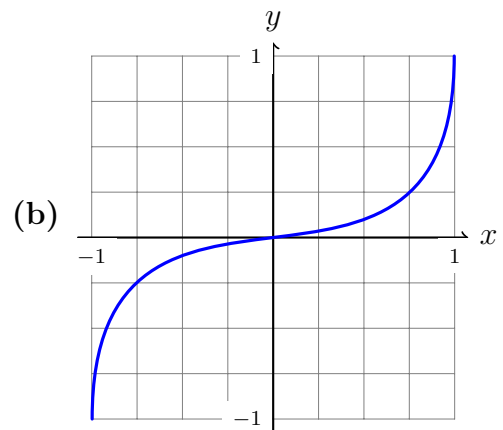
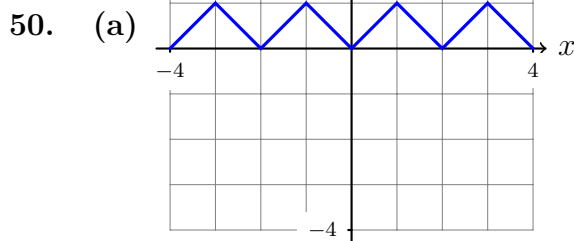
49. (a) *Worked Solution:*

$$f'(0) = \lim_{h \rightarrow 0} \frac{h^{\frac{3}{5}}}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{\frac{2}{5}}}$$

which does not exist because as h approaches 0, $h^{\frac{2}{5}}$ approaches 0 but 1 does not. So $f'(0)$ does not exist.

(b) $f'(0)$ exists.

(c) $f'(0)$ exists.



51. It follows that f' is an odd function, because

$$\begin{aligned}
 f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{-h} \\
 &= - \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= -f'(x).
 \end{aligned}$$

Here the third equation came from substituting h for $-h$ in the limit. (Why is this valid?)

52. (a) $x^{1/n}$ (b) x^{ab} (c) x^{a+b} (d) x^{-a} (e) $x^{7/12}$ (f) $x^{-17/6}$

53. (a) 0. (b) 0.

(c) *Worked Solution:* $\frac{d}{dx}[3x+2] = \frac{d}{dx}[3x] + \frac{d}{dx}[2] = 3\frac{d}{dx}[x] + \frac{d}{dx}[2] = 3 \cdot 1 + 0 = 3.$

(d) $-77x^{10}.$ (e) $6x^2 - 2x.$

(f) *Worked Solution:* $\frac{d}{dt}[f(t)] = \frac{d}{dt}\left[\frac{1}{3}t^5 - 2t^3 + t^2\right] = \frac{d}{dt}\left[\frac{1}{3}t^5\right] - \frac{d}{dt}[2t^3] + \frac{d}{dt}[t^2] = \frac{1}{3}\frac{d}{dt}[t^5] - 2\frac{d}{dt}[t^3] + \frac{d}{dt}[t^2] = \frac{1}{3}5t^4 - 2 \cdot 3t^2 + 2t = \frac{5}{3}t^4 - 6t^2 + 2t.$

(g) $-\frac{3}{5}x^{-8/5}.$ (h) $-3cy^{-4}.$

(i) *Worked Solution:* $\frac{d}{ds}[B(s)] = \frac{d}{ds}\left[\frac{8}{s^3}\right] = \frac{d}{ds}[8s^{-3}] = 8\frac{d}{ds}[s^{-3}] = 8 \cdot (-3)s^{-4} = -24s^{-4}.$

(j) $\frac{4}{3}r^{1/3} - \frac{1}{3}r^{-2/3}$ (k) $1 + \frac{1}{\sqrt{p}}$ (l) $2\pi r$ (m) $\sqrt{3}\left(1 - \frac{1}{2\sqrt{u}}\right)$

(n) $\frac{1}{3}u^{-2/3} + \frac{9}{2}u^{1/2}$

54. Both methods give $f'(x) = 8x^3 + 6x^2 - 2x - 1$. I found it quicker to multiply first and then differentiate.

55. Both methods give $f'(x) = 2 - \frac{3}{2}x^{-5/2} + x^{-2}$. It is faster to simplify first.

56. (a) $6x^2 - 4x^3$ (b) $4x + 1$ (c) $\frac{8}{3}x^{1/3} + x^{-2/3}$ (d) $8t - 4$

(e) *Worked Solution:* By multiplying out $(x - x^{-1})(x - x^{-1})(x - x^{-1})$, or by using the binomial theorem, we find that $H(x) = x^3 - 3x + 3x^{-1} - x^{-3}$, so that $H'(x) = 3x^2 - 3 - 3x^{-2} + 3x^{-4}$.

(f) $1 - x^{-2}$ (g) $-10y^{-3} - 9y^{-2}$ (h) $-3s^{-2} - 9s^{-4} - 2s^{-3} - 4s^{-5}$

(i) $\frac{2t + t^4}{(1 - t^3)^2}$ (j) $-2t^{-3} - 1$

(k) *Worked Solution:*

$$f(x) = \frac{x+2}{x^2+3x+2} = \frac{x+2}{(x+2)(x+1)} = \frac{1}{x+1}$$

and so by the quotient rule

$$f'(x) = \frac{(x+1) \cdot 0 - (1+0) \cdot 1}{(x+1)^2} = -\frac{1}{(x+1)^2}.$$

(l) $1 - v^{-2/3}$ (m) $\frac{1}{2}t^{-1/2} - \frac{1}{6}t^{-7/6}$ (n) $\frac{1}{2\sqrt{t}}$ (o) $\frac{x^4 - 3x^2}{(x^2 - 1)^2}$
 (p) $2ax + b$

57. (a) *Worked Solution:* $y = \frac{x}{2x-1}$, so

$$\frac{dy}{dx} = \frac{(2x-1) \cdot 1 - x \cdot (2-0)}{(2x-1)^2} = \frac{2x-1-2x}{(2x-1)^2} = \frac{-1}{(2x-1)^2}.$$

So when $x = 1$ we have $\frac{dy}{dx} = \frac{-1}{(2 \cdot 1 - 1)^2} = -1$. Thus the equation of the tangent line through $(1, 1)$ is $y - 1 = -1(x - 1)$ or in other words $y = -x + 2$.

(b) $y = 7x - 3$

58. (a) $f'(x) = 4x^3 - 6x^2 + 6x - 4$ and $f''(x) = 12x^2 - 12x + 6$.

(b) *Worked Solution:* Since $G(r) = \sqrt[4]{r} - 2\sqrt[3]{r} = r^{1/4} - 2r^{1/3}$, we have

$$G'(r) = \frac{1}{4}r^{\frac{1}{4}-1} - 2 \cdot \frac{1}{3}r^{\frac{1}{3}-1} = \frac{1}{4}r^{-\frac{3}{4}} - \frac{2}{3}r^{-\frac{2}{3}}$$

and

$$G''(r) = \frac{1}{4} \cdot \left(-\frac{3}{4}\right)r^{-\frac{3}{4}-1} - \frac{2}{3} \cdot \left(-\frac{2}{3}\right)r^{-\frac{2}{3}-1} = -\frac{3}{16}r^{-\frac{7}{4}} + \frac{4}{9}r^{-\frac{5}{3}}$$

(c) $f'(x) = \frac{6x^2 + 2x^3}{(2+x)^2}$ and $f''(x) = 2x \frac{x^2 + 6x + 12}{(x+2)^3}$.

(d) $f'(x) = \frac{3}{(2-3x)^2}$ and $f''(x) = \frac{18}{(2-3x)^3}$.

(e) *Worked Solution:* Since $f(x) = \frac{1}{x^2 - x - 1}$ we have by the chain rule:

$$\begin{aligned} f'(x) &= -\frac{1}{(x^2 - x - 1)^2} \cdot (2x - 1) \\ &= -\frac{2x - 1}{(x^2 - x - 1)^2}. \end{aligned}$$

Then by the quotient rule we have:

$$\begin{aligned}
 f''(x) &= -\frac{(x^2 - x - 1)^2 \cdot 2 - (2x - 1) \cdot 2 \cdot (2x - 1)(x^2 - x - 1)}{(x^2 - x - 1)^4} \\
 &= -\frac{(x^2 - x - 1) \cdot 2 - (2x - 1) \cdot 2 \cdot (2x - 1)}{(x^2 - x - 1)^3} \\
 &= -\frac{(2x^2 - 2x - 2) - 2(4x^2 - 4x + 1)}{(x^2 - x - 1)^3} \\
 &= -\frac{2x^2 - 2x - 2 - 8x^2 + 8x - 2}{(x^2 - x - 1)^3} \\
 &= -\frac{-6x^2 + 6x - 4}{(x^2 - x - 1)^3} \\
 &= 2 \cdot \frac{3x^2 - 3x + 2}{(x^2 - x - 1)^3}
 \end{aligned}$$

59. (a) $h'(2) = 3$.

(b) *Worked Solution:* Since $h(x) = f(x)g(x)$, the product rule shows us that $h'(x) = f'(x)g(x) + f(x)g'(x)$. So $h'(2) = f'(2)g(2) + f(2)g'(2) = (-1) \cdot 2 + 3 \cdot 0 = -2$.

(c) $h'(2) = 2/9$.

(d) $h'(2) = -1/2$.

(e) *Worked Solution:* Since $h(x) = \frac{f(x)}{1-g(x)}$, the quotient rule gives

$$\begin{aligned}
 h'(x) &= \frac{d}{dx} \left(\frac{f(x)}{1-g(x)} \right) \\
 &= \frac{\frac{d}{dx}[f(x)] \cdot (1-g(x)) - f(x) \cdot \frac{d}{dx}[1-g(x)]}{(1-g(x))^2} \\
 &= \frac{f'(x)(1-g(x)) + f(x)g'(x)}{(1-g(x))^2}.
 \end{aligned}$$

Thus

$$h'(2) = \frac{f'(2)(1-g(2)) + f(2)g'(2)}{(1-g(2))^2} = \frac{(-1) \cdot (1-2) + 3 \cdot 0}{(1-2)^2} = 1.$$

(f) *Worked Solution:* $(q \circ r)'(1) = q'(r(1)) \cdot r'(1) = q'(2) \cdot r'(1) = 9 \cdot 4 = 36$.

(g) *Worked Solution:* We write $p \circ q \circ r$ as $p \circ (q \circ r)$ and apply the chain rule to get

$$(p \circ q \circ r)'(x) = (p \circ (q \circ r))'(x) = p'((q \circ r)(x)) \cdot (q \circ r)'(x) = p'(q(r(x))) \cdot (q \circ r)'(x).$$

Now to find $(q \circ r)'(x)$ we may either use what we were told in the previous part, or we apply the chain rule again, as follows.

$$(p \circ q \circ r)'(x) = p'(q(r(x))) \cdot (q \circ r)'(x) = p'(q(r(x))) \cdot q'(r(x)) \cdot r'(x)$$

as required. Thus

$$(p \circ q \circ r)(1) = p'(q(r(1))) \cdot q'(r(1)) \cdot r'(1) = p'(q(2)) \cdot q'(2) \cdot 4 = p'(3) \cdot 9 \cdot 4 = 4 \cdot 9 \cdot 4 = 144.$$

61. We have $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. So if $f'(x) = 3$ we get $x^{\frac{2}{3}} = \frac{1}{9}$. This equation has the unique solution $x = \frac{1}{27}$. We then get $f(x) = \frac{1}{3}$ so the equation of the tangent line is $y - \frac{1}{3} = 3(x - \frac{1}{27})$ or $y = 3x + \frac{2}{9}$.

63. (a) The function is differentiable for all $a \neq \pm 1$.

(b) The function is differentiable for $x \neq 1$ In both cases we prove this by calculating the limit in the definition of the derivative from left and from right.

64. (a) *Worked Solution:*

$$\lim_{\theta \rightarrow 0} \frac{\theta}{\sin(\theta)} = \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin(\theta)}{\theta}} = \frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta}} = \frac{1}{1} = 1.$$

(b) $\lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin(\theta)} = 0$.

(c) *Worked Solution:* We divide top and bottom of the fraction by θ so that the expression now consists of terms whose limits we know.

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta + \sin(\theta)} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin(\theta)}{\theta}}{1 + \frac{\sin(\theta)}{\theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta}}{1 + \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta}} = \frac{1}{1 + 1} = 1/2$$

(d) $\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta + \theta^2} = 1$.

65. (a) $f'(x) = 2 \cos x - 3x^2$

(b) $f'(x) = \cos x - x \sin x$

(c) $f'(x) = -\sin x - \frac{1}{2} \csc x \cot x$

(d) $\frac{dy}{dx} = -2 \csc x \cot x - \sec x \tan x$

(e) $g'(t) = 2t \sin t + t^2 \cos t$

(f) $g'(t) = \sec t [\sec t - 2 \tan t]$

(g) *Worked Solution:* Since $y = a \sin t + t^2 \cos t$, we have

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt} [a \sin t + t^2 \cos t] \\ &= a \frac{d}{dt} [\sin t] + \frac{d}{dt} [t^2 \cos t] \\ &= a \cos t + t^2 \frac{d}{dt} [\cos t] + \frac{d}{dt} [t^2] \cdot \cos t \\ &= a \cos t - t^2 \sin t + 2t \cos t. \end{aligned}$$

(h) $y' = (a \sin u + b \tan u) + u(a \cos u + b \sec^2 u)$

(i) $y' = \frac{2 - \cot x + x \csc^2 x}{(2 - \cot x)^2}$

(j) $\frac{dy}{dx} = \sec x \cdot [\tan^2 x + \sec^2 x]$

$$(k) f'(\theta) = \frac{\sec^2 \theta}{(1 + \tan \theta)^2}$$

(l) *Worked Solution:*

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\frac{\cos x}{1 + \sin x} \right] \\ &= \frac{(1 + \sin x) \cdot \frac{d}{dx} \cos x - \cos x \cdot \frac{d}{dx} (1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{-(1 + \sin x) \cdot \sin x - \cos x \cdot \cos x}{(1 + \sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-\sin x - 1}{(1 + \sin x)^2} \\ &= -\frac{1}{1 + \sin x} \end{aligned}$$

$$(m) y' = -\frac{\sec x + \sec^2 x}{\tan^2 x}$$

$$(n) \frac{dy}{dx} = x \cos(x) \cot(x) [2 + x \cot(x)] \quad x \cos x [2 \cot x - x - x \csc^2 x]$$

66. *Worked Solution:*

$$\begin{aligned} \frac{d}{dx} \cos x &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left[\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right] \\ &= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x. \end{aligned}$$

67. *Worked Solution for sec x:*

$$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = -\frac{\frac{d}{dx} \cos x}{\cos^2 x} = -\frac{-\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

68. (a) $y = \frac{2}{3}x + (\frac{2}{\sqrt{3}} - \frac{\pi}{9})$

(b) $y = x$

(c) *Worked Solution:* Since $y = x + \sec x$, we have $\frac{dy}{dx} = \frac{d}{dx}[x + \sec x] = 1 + \sec x \tan x$, so when $x = \pi$ we have $\frac{dy}{dx} = 1 + \sec \pi \tan \pi = 1 + -1 \cdot 0 = 1$. Thus the equation of the tangent line is $y - (\pi - 1) = 1(x - \pi)$, or in other words $y = x - 1$.

69. *Worked Solution:*

Since $45 = 4 \times 11 + 1$, we have $\frac{d^{45}}{dx^{45}} \sin x = \frac{d}{dx} \sin x = \cos x$.

For the second part, we work out that $\frac{d^4}{dx^4}[x \cos x] = \frac{d^3}{dx^3}[-x \sin x + \cos x] = \frac{d^2}{dx^2}[-x \cos x - 2 \sin x] = \frac{d}{dx}[x \sin x - 3 \cos x] = x \cos x + 4 \sin x$. Thus $\frac{d^8}{dx^8}[x \cos x] = \frac{d^4}{dx^4}[x \cos x + 4 \sin x] = x \cos x + 4 \sin x + 4 \sin x = x \cos x + 8 \sin x$. Proceeding in this way, we find that $\frac{d^{16}}{dx^{16}}[x \cos x] = x \cos x + 16 \sin x$.

70. It has horizontal tangent at the points of the form $(\frac{2\pi}{3} + 2k\pi, \frac{1}{\sqrt{3}})$ and $(-\frac{2\pi}{3} + 2k\pi, -\frac{1}{\sqrt{3}})$.

71. (a) *Worked Solution:* $y = \cos(3x) = \cos(g(x)) = f(g(x))$ where $g(x) = 3x$ and $f(u) = \cos(u)$.

Consequently $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$, and since $f'(u) = -\sin(u)$ and $g'(x) = 3$, we have

$$\frac{dy}{dx} = -\sin(g(x)) \cdot 3 = -3 \sin(3x).$$

(b) $g(x) = 4 + 3x$, $f(u) = \sqrt{u}$, $\frac{dy}{dx} = \frac{3}{2\sqrt{4+3x}}$

(c) $g(x) = 1 - x^3$, $f(u) = u^5$, $\frac{dy}{dx} = -15x^2(1 - x^3)^4$.

(d) *Worked Solution:* $y = \sqrt[3]{\sin(x)} = \sqrt[3]{g(x)} = f(g(x))$ where $g(x) = \sin(x)$ and $f(u) = \sqrt[3]{u}$.

So $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$. Now $f'(u) = \frac{1}{3u^{2/3}}$ and $g'(x) = \cos(x)$, so that

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = \frac{1}{3(g(x))^{2/3}} \cdot \cos(x) = \frac{\cos(x)}{3(\sin(x))^{2/3}}.$$

72. (a) $F'(x) = 16x(x^4 - 2x^2 + 2)^3(x^2 - 1)$

(b) $F'(x) = \frac{1 - x^2}{(1 + 3x - x^3)^{2/3}}$

(c) $g'(t) = -\frac{12t^2}{(t^3 + 1)^5}$

(d) $\frac{dy}{dx} = 4x^3 \cos(a^4 + x^4)$

(e) $\frac{dy}{dx} = 4 \sin^3 x \cdot \cos x$

$$(f) y' = x \csc(kx)[2 - kx \cot(kx)].$$

$$(g) f'(x) = (3x - 2)^3(x^4 - x - 1)^4[72x^4 - 40x^3 - 27x - 2]$$

$$(h) h'(t) = [34t^2 - 32t - 1] \frac{(2t^2 - 1)^3}{2(t - 1)^{1/2}}$$

$$(i) y' = 12x \frac{(x^2 - 1)^2}{(x^2 + 1)^4}$$

$$(j) y' = -\sin(x \sin x)(x \cos x + \sin x)$$

$$(k) F'(z) = -\frac{1}{(z + 1)^{1/2}(z - 1)^{3/2}}$$

$$(l) y' = \frac{1}{(1 - r^2)^{3/2}}$$

$$(m) y' = \frac{x \sin \sqrt{1 - x^2}}{\sqrt{1 - x^2}}$$

$$(n) y' = 2 \cos(2x) \cdot \cos^{-2}(\sin 2x)$$

$$(o) y' = -16 \cos 2x \frac{(1 - \sin 2x)^3}{(1 + \sin 2x)^5}$$

$$(p) y' = \frac{2x^{1/2} - 1}{4x^{1/2}(x - x^{1/2})^{1/2}}$$

73. (a) *Worked Solution:* The first derivative is

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x = 2x \cos(x^2)$$

and the second derivative is

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}(2x) \cdot \cos(x^2) + 2x \cdot \frac{d}{dx} \cos(x^2) \\ &= 2 \cos(x^2) + 2x \cdot 2x \cdot (-\sin(x^2)) \\ &= 2 \cos(x^2) - 4x^2 \sin(x^2). \end{aligned}$$

$$(b) \frac{dy}{dx} = 2 \sin(x) \cos(x) \text{ and } \frac{d^2y}{dx^2} = 2(\cos^2(x) - \sin^2(x)).$$

$$(c) K'(t) = 5 \sec^2(5t) \text{ and } K''(t) = 50 \sec^2(5t) \tan(5t)$$

74. *Worked Solution:* The product rule tells us that

$$f'(x) = \frac{d}{dx}(x) \cdot g(x^3) + x \cdot \frac{d}{dx}(g(x^3)) = g(x^3) + x \frac{d}{dx}(g(x^3)),$$

and the chain rule tells us that $\frac{d}{dx}(g(x^3)) = g'(x^3) \cdot 3x^2$. So

$$f'(x) = g(x^3) + 3x^3 g'(x^3).$$

Similarly,

$$f''(x) = 3x^2 g'(x^3) + 9x^2 g'(x^3) + 3x^3 \cdot 3x^2 \cdot g''(x^3) = 12x^2 g'(x^3) + 9x^5 g''(x^3).$$

75. (a) $y' = -\frac{x^4}{y^4}$

(b) *Worked Solution:* Differentiating both sides of the equation $4\sqrt{x} - 4\sqrt{y} = 1$ yields

$$4\frac{d}{dx}\sqrt{x} - 4\frac{d}{dx}\sqrt{y} = \frac{d}{dx}1$$

i.e.

$$4\frac{1}{\sqrt{x}} - 4\frac{1}{\sqrt{y}}\frac{d}{dx}y = 0$$

i.e.

$$\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}y' = 0.$$

Rearranging this gives

$$y' = \frac{\sqrt{y}}{\sqrt{x}}.$$

(c) $y' = -\frac{2x + y}{x + 2y}$

(d) $y' = -\frac{3x^2 + 4xy - 2y^3}{2x^2 - 6xy^2}$

(e) *Worked Solution:* The given equation $x^3(x + y) = y^3(2x - y)$ can be written as $x^4 + x^3y = 2xy^3 - y^4$. Differentiating both sides with respect to x gives

$$4x^3 + (3x^2y + x^3y') = (2y^3 + 2x \cdot 3y^2y') - 4y^3y'.$$

Rearranging,

$$4x^3 + 3x^2y + x^3y' = 2y^3 + 6xy^2y' - 4y^3y'.$$

$$4x^3 + 3x^2y - 2y^3 = 6xy^2y' - 4y^3y' - x^3y'$$

$$4x^3 + 3x^2y - 2y^3 = y'(6xy^2 - 4y^3 - x^3)$$

so that

$$y' = \frac{4x^3 + 3x^2y - 2y^3}{6xy^2 - 4y^3 - x^3}.$$

(f) $y' = -\frac{y^2 + 2x \sin y}{2xy + x^2 \cos y}$

(g) $y' = \frac{1 - y^2 \sin(xy^2)}{2xy \sin(xy^2)}$

(h) $y' = -\frac{\tan x}{\tan y}$

(i) $y' = \frac{\cos y^2 + 2xy \sin x^2}{\cos x^2 + 2xy \sin y^2}$

(j) $y' = \frac{y(\cos(x/y) - y^2)}{x(y^2 + \cos(x/y))}$

(k) Worked Solution: Differentiating both sides of the equation $\sqrt{x-y} = 1 - x^2y^2$ with respect to x gives

$$\frac{1}{2\sqrt{x-y}} \frac{d}{dx}(x-y) = -\frac{d}{dx}(x^2y^2)$$

or in other words

$$\frac{1}{2\sqrt{x-y}}(1-y') = -(2x \cdot y^2 + x^2 \cdot 2yy')$$

Rearranging gives the following:

$$\begin{aligned} \frac{1}{2\sqrt{x-y}} - \frac{y'}{2\sqrt{x-y}} &= -2xy^2 - 2x^2yy' \\ \frac{1}{2\sqrt{x-y}} + 2xy^2 &= \frac{y'}{2\sqrt{x-y}} - 2x^2yy' \\ \frac{1}{2\sqrt{x-y}} + 2xy^2 &= y' \left(\frac{1}{2\sqrt{x-y}} - 2x^2y \right) \end{aligned}$$

Thus

$$y' = \frac{\frac{1}{2\sqrt{x-y}} + 2xy^2}{\frac{1}{2\sqrt{x-y}} - 2x^2y} = \frac{1 + 4xy^2\sqrt{x-y}}{1 - 4x^2y\sqrt{x-y}}$$

$$(l) y' = \frac{y - 2y^{5/2}x^{1/2}}{4x^{3/2}y^{3/2} - x}$$

$$(m) y' = \frac{\cos y + y \sin x}{\cos x + x \sin y}$$

76. (a) $f'(1) = 2/3$

(b) Worked Solution: Differentiating both sides of the equation $g(x) + x \cos(g(x)) = x^3$ gives $g'(x) + \cos(g(x)) - x \sin(g(x))g'(x) = 3x^2$, and setting $x = 0$ gives $g'(0) + \cos(g(0)) = 0$. We can compute $g(0)$ by taking the original equation $g(x) + x \cos(g(x)) = x^3$ and setting $x = 0$ to give $g(0) = 0$, so that $\cos(g(0)) = 1$. It follows that $g'(0) = -1$.

77. (a) Worked Solution: Differentiating both sides of the equation $y \cos(2x) = x \sin(2y)$ gives

$$y' \cos(2x) - 2y \sin(2x) = \sin(2y) + 2x \cos(2y)y'$$

Setting $x = \pi/4$ and $y = \pi/2$ gives

$$y' \cos(\pi/2) - \pi \sin(\pi/2) = \sin(\pi) + (\pi/2) \cos(\pi)y'$$

or in other words

$$-\pi = (-\pi/2)y'$$

so that $y' = 2$. Thus the gradient of the tangent line to the curve through the point $(\pi/4, \pi/2)$ is 2, and so the gradient of the tangent is

$$y - \pi/2 = 2(x - \pi/4)$$

or in other words $y = 2x$.

(b) $y = -3x + \pi$

(c) $y = x - 2$

(d) $y = -\sqrt{3}x + 4\sqrt{3}$

(e) $y = \frac{5\sqrt{5}}{8}x - \frac{9}{8}$

78. (a) *Worked Solution:* Differentiating both sides of the equation

$$x^2 + 4y^2 = 4$$

and rearranging gives

$$y' = -\frac{x}{4y}.$$

Differentiating this expression then gives

$$y'' = \frac{-4y + 4xy'}{16y^2}.$$

Substituting our expression for y' now gives

$$y'' = \frac{-4y + 4x \frac{-x}{4y}}{16y^2} = \frac{-4y^2 - x^2}{16y^3}.$$

We may now use the original equation to obtain

$$y'' = \frac{-1}{4y^3}.$$

(b) $y'' = \frac{\sqrt{xy} - y}{2x\sqrt{xy}}$

(c) *Worked Solution:* Differentiating the equation

$$x^5 + y^5 = 1$$

gives

$$5x^4 + 5y^4y' = 0$$

which rearranges to give

$$y' = -\frac{x^4}{y^4}.$$

Differentiating this expression gives

$$\begin{aligned} y'' &= -\frac{y^4 \cdot 4x^3 - x^4 \cdot 4y^3 y'}{y^8} \\ &= -\frac{y^4 \cdot 4x^3 - x^4 \cdot 4y^3 \cdot (-x^4/y^4)}{y^8} \\ &= -\frac{4x^3 y^4 + 4x^8 y^{-1}}{y^8} \\ &= -4 \frac{x^3 y^5 + x^8}{y^9} \\ &= -4 \frac{x^3}{y^9} (y^5 + x^5). \end{aligned}$$

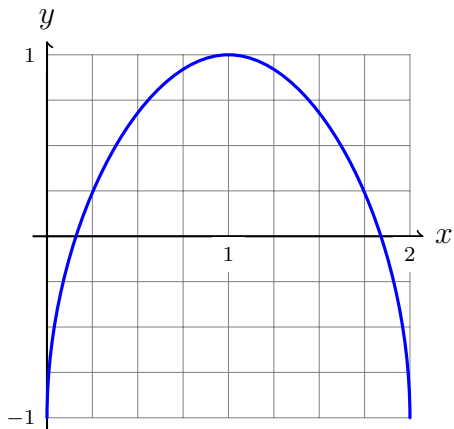
Now using the original equation we find that

$$y'' = -4 \frac{x^3}{y^9}.$$

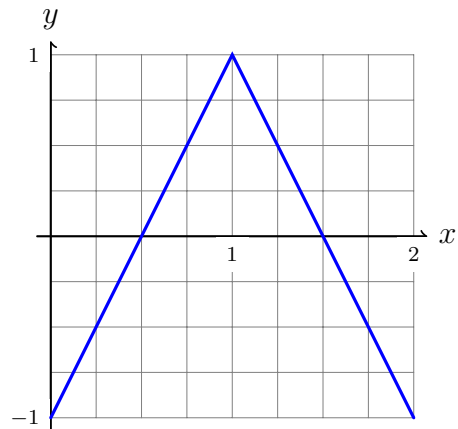
(d) $y'' = \frac{-2xy}{(3y^2 - x)^3}$

79.

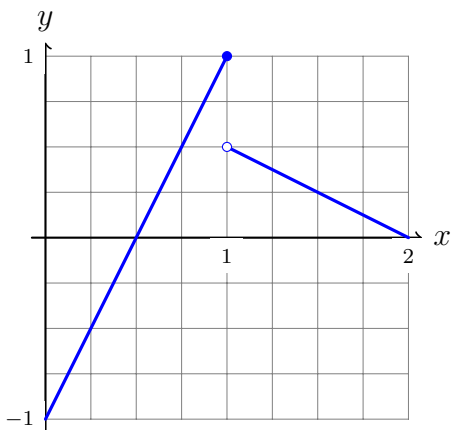
(a)



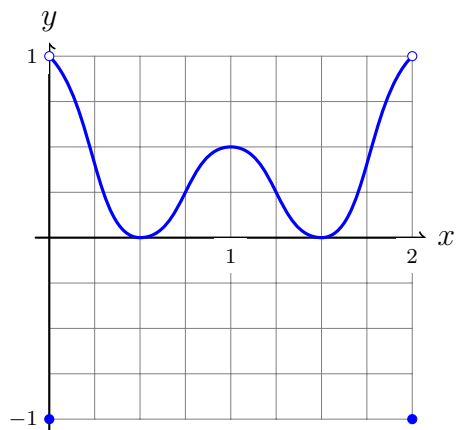
(b)



(c)



(d)



80. (a) $-1/4$

(b) 1

(c) *Worked Solution:* The function is continuous and its domain is \mathbb{R} .

For $t > 3/4$ we have $4t - 3 > 0$, and so $g(t) = 4t - 3$ and consequently $g'(t) = 4 \neq 0$. So $g'(t)$ exists and is nonzero for all $t > 3/4$, and consequently there are no critical numbers in this range.

For $t < 3/4$ we find, similarly, that there are no critical numbers.

For $t = 3/4$ we find that $g'(t)$ does not exist, since (as can be checked) $\lim_{h \rightarrow 0^+} \frac{g(3/4+h)-g(3/4)}{h}$ and $\lim_{h \rightarrow 0^-} \frac{g(3/4+h)-g(3/4)}{h}$ exist but are not equal. So $t = 3/4$ is a critical number.

The only critical number is $3/4$.

(d) 0, -2 .

(e) *Worked Solution:* The domain of h is $[0, \infty)$. Now $h'(t) = \frac{t^{-3/4}}{4}(6t^{1/2} - 1)$, and in particular it exists for all t in the domain of h , except for $t = 0$. Thus the critical numbers are those t such that $h'(t) = 0$, i.e. $t = 1/36$, and 0.

(f) The critical numbers are 0, where $F'(x)$ does not exist, and 3 and $1/2$, where $F'(x) = 0$.

81. (a) The absolute maximum is $f(3) = 9$ and the absolute minimum is $f(1) = 1$.

(b) The absolute maximum is $f(0) = 2$. and the absolute minimum is $f(2) = -14$.

(c) The absolute maximum is $f(2) = 187$ and the absolute minimum is $f(0) = -5$.

(d) *Worked Solution:* We will use the closed interval method. This can be applied because the function f is a product of a polynomial with a square root of a polynomial, and so is continuous on the whole domain $[-1, 3]$.

0. $f'(t) = \sqrt{9-t^2} + \frac{t}{2\sqrt{9-t^2}}(-2t) = \frac{1}{9-t^2}(9-2t^2)$. This is defined for all $t \in [-1, 3)$, but not for $t = 3$. So the critical numbers are $t = 3$ and those $t \in [-1, 3)$ for which $f'(t) = 0$, i.e. $3/\sqrt{2}$.

1. The only critical number in the interior $(-1, 3)$ is $3/\sqrt{2}$, and $f(3/\sqrt{2}) = 3/\sqrt{2}\sqrt{9-9/2} = 9/2$.

2. $f(-1) = -2\sqrt{2}$ and $f(3) = 0$.

So the absolute maximum and minimum are the largest and smallest numbers from parts 1 and 2. In other words the absolute maximum is $f(3/\sqrt{2}) = 9/2$ and the absolute minimum is $f(-1) = -2\sqrt{2}$.

(e) The absolute maximum is $f(\pi/2) = 3$ and the absolute minimum is $f(7\pi/6) = f(11\pi/6) = -1.5/$

82. (a) The only value satisfying the conclusion is $c = 1$.

(b) *Worked Solution:* f is continuous on the interval $[0, 2]$ because $x \geq 0$ for all $x \in [0, 2]$, and f is differentiable on this interval because $x > 0$ for all $x \in (0, 2)$. So the hypotheses of the mean value theorem hold.

Next, for the conclusions of the theorem to hold for $c \in (0, 2)$ means that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

Since $f'(x) = \frac{1}{2\sqrt{x}}$, we see that the conclusions hold for c if $\frac{1}{2\sqrt{c}} = \frac{1}{\sqrt{2}}$, so the conclusions hold if and only if $c = \frac{1}{2}$.

- 83.** *Worked Solution:* Let f be the function defined by $f(x) = 2x + \sin(x)$. Then $f(2\pi) = 4\pi + 0 = 4\pi > 0$ and $f(-2\pi) = -4\pi + 0 = -4\pi$. Thus $f(2\pi) > 0 > f(-2\pi)$, and f is continuous on the interval $[-2\pi, 2\pi]$, so by the intermediate value theorem there is $x_0 \in (-2\pi, 2\pi)$ such that $f(x_0) = 0$. In other words the equation has a root x_0 .

Suppose that the equation has a second root x_1 . Then since $f(x_0) = f(x_1) = 0$ and f is differentiable everywhere, Rolle's theorem shows that there is $c \in (x_0, x_1)$ such that $f'(c) = 0$. However $f'(x) = 2 + \cos(x) \geq 2 - 1 = 1 > 0$ so that no such c exists, and so no second root could have existed.

We have shown that there is at least one root, but there cannot be two. So there is exactly one root as required.

- 84.** (a) *Worked Solution:* We will show that it is impossible for the equation to have two roots in $[-2, 2]$. Suppose it does, call them x_0 and x_1 , and (by swapping them if necessary) assume that $-2 \leq x_0 < x_1 \leq 2$.

Let f be the function defined by $f(x) = x^3 - 14x + 5$. Then f is a polynomial, so is continuous on $[x_0, x_1]$ and is differentiable on (x_0, x_1) . (In fact it is continuous and differentiable everywhere.) Then $f(x_0) = f(x_1) = 0$, so that by Rolle's Theorem there is $c \in (x_0, x_1)$ such that $f'(c) = 0$. Now by computing f' we see that $f'(c) = 3c^2 - 14$. However, since $c \in (-2, 2)$, we have $f'(c) = 3c^2 - 14 < 3 \times 4 - 14 = -2$, so that $f'(c) = 0$ is impossible.

This contradiction means that there could not have been two roots in the first place.

- 85.** (a) *Worked Solution:* Since f is differentiable, it satisfies the assumptions of the Mean Value Theorem for the interval $[-1, 2]$. Thus there is $c \in (-1, 2)$ for which

$$\frac{f(2) - f(-1)}{2 - (-1)} = f'(c)$$

or in other words $f(2) - f(-1) = 3f'(c)$. Since $2 \leq f'(x) \leq 4$, it follows that $6 \leq f(2) - f(-1) \leq 12$.

- 86.** Define h by $h(x) = g(x) - f(x)$. The given inequalities show that $h(a) = g(a) - f(a) \geq 0$ and that $h'(x) = g'(x) - f'(x) \geq 0$ for $x \in (a, b)$. And we want to show that $h(b) \geq 0$ since then $g(b) - f(b) \geq 0$ so that $g(b) \geq f(b)$.

Since f and g are continuous on $[a, b]$ and differentiable on (a, b) , the same is true for the difference h . So we may apply the Mean Value Theorem: there is $c \in (a, b)$ such that

$$\frac{h(b) - h(a)}{b - a} = h'(c)$$

and consequently $h(b) = (b-a)h'(c) + h(a)$. Since $a < b$ we have $(b-a) > 0$, and since $c \in (a, b)$ we have $h'(c) \geq 0$, so that $(b-a)h'(c) \geq 0$. And we know that $h(a) \geq 0$, so that $(b-a)h'(c) + h(a) \geq 0$. Thus $h(b) \geq 0$ as required.

- 87. (a) Worked Solution:** We will use the fact that the local maxima or minima of f are critical points of f .

The function is continuous and its derivative exists for all x , so its critical points are the points where $f'(x) = 0$. Now $f'(x) = 3x^2 - 3$, which is zero for $x = 1$ and $x = -1$. So the possible local maxima and minima occur at $x = \pm 1$.

To determine if they are local maxima or minima, we use the second derivative test, which applies since $f''(x)$ exists and is continuous for all x , and is given by $f''(x) = 6x$.

Since $f''(1) = 6 > 0$, f has a local minimum at 1.

Since $f''(-1) = -6 < 0$, f has a local maximum at -1 .

(b) Worked Solution: We will use the fact that the local maxima and minima are all critical points. The domain of the function is \mathbb{R} and it is continuous everywhere, so that the critical points are the points where $f'(x) = 0$ or where $f'(x)$ does not exist.

The derivative is $f'(x) = 1 + \frac{1}{3}x^{-2/3}$, which exists for all $x \in \mathbb{R}$ except for $x = 0$. So $x = 0$ is a critical number. There are no other critical numbers, since $f'(x) > 0$ for all $x \neq 0$. So the only possible local maximum or minimum is at $x = 0$. Since $f'(x)$ does not exist there, we cannot use the second derivative test. But the first derivative test applies, and indeed, since $f'(x) > 0$ for all $x \neq 0$, it tells us that 0 is neither a local maximum nor a local minimum.

So f has no local maxima or minima.

(c) f has a local minimum at 0 and a local maximum at -2 .

- 88. (a)** Increasing on $(-\infty, -2)$ and $(3, \infty)$, decreasing on $(-2, 3)$, local maximum at $(-2, 44)$, local minimum at $(3, -81)$.

(b) Worked Solution: First,

$$f'(x) = 12x^2 - 18x + 6 = 6(2x^2 - 3x + 1) = 6(2x - 1)(x - 1).$$

Thus $f'(x) = 0$ if and only if $x = 1/2$ or $x = 1$. The sign of $f'(x)$ can be computed as follows:

- For $x < 1/2$ we have $(2x - 1) < 0$ and $(x - 1) < 0$, so that $f'(x) > 0$.
- For $1/2 < x < 1$ we have $(2x - 1) > 0$ and $(x - 1) < 0$, so that $f'(x) < 0$.
- For $1 < x$ we have $(2x - 1) > 0$ and $(x - 1) > 0$ so that $f'(x) > 0$.

So f is increasing on $(-\infty, 1/2)$ and $(1, \infty)$ and it is decreasing on $(1/2, 1)$.

As above, the critical numbers of f are $x = 1/2$ and $x = 1$. Now $f(1/2) = 9/4$ and $f(1) = 2$ so that the critical points are $(1/2, 9/4)$ and $(1, 2)$. And we can compute

$$f''(x) = 24x - 18$$

so that $f''(1/2) = -6 < 0$ and $f''(1) = 6 > 0$ and consequently $(1/2, 9/4)$ is a local maximum and $(1, 2)$ is a local minimum.

(c) f is decreasing on $(-\infty, -1)$ and $(0, 1)$, it is increasing on $(-1, 0)$ and $(1, \infty)$, there is a local maximum at $(0, -3)$ and there are local minima at $(-1, -4)$ and $(1, -4)$.

(d) f is increasing on $(-\infty, -\sqrt{2})$ and on $(-\sqrt{2}, 0)$. f is decreasing on $(0, \sqrt{2})$ and on $(\sqrt{2}, \infty)$, and there is a local maximum at $(0, 0)$.

(e) *Worked Solution:* First we compute $f'(x) = 1 - 2\sin(x)$, so that $f'(x) = 0$ when $\sin(x) = 1/2$, or in other words when $x = \pi/6$ and $x = 5\pi/6$. Now by considering the definition of $\sin(x)$ we see that:

- For $0 \leq x < \pi/6$ we have $f'(x) > 0$;
- for $\pi/6 < x < 5\pi/6$ we have $f'(x) < 0$;
- for $5\pi/6 < x \leq 2\pi$ we have $f'(x) > 0$.

So f is increasing on $[0, \pi/6)$ and $(5\pi/6, 2\pi]$, and it is decreasing on $(\pi/6, 5\pi/6)$.

Next,

$$f''(x) = -2\cos(x),$$

so that $f''(\pi/6) = -2\cos(\pi/6) = -\sqrt{3} < 0$ and $f''(5\pi/6) = -2\cos(5\pi/6) = \sqrt{3} > 0$ so that there is a local maximum at $(\pi/6, \pi/6 + \sqrt{3})$ and a local minimum at $(5\pi/6, 5\pi/6 - \sqrt{3})$.

(f) f is increasing on $(4, \infty)$ and it is decreasing on $[0, 4)$. It has a local minimum at $(4, -4)$.

89. (a) *Worked Solution:* The domain of $f(x) = \frac{1-e^{x^2}}{1-e^{4-x^2}}$ is the set of x such that $1-e^{4-x^2} \neq 0$, or equivalently $e^{4-x^2} \neq 1$, or equivalently $4-x^2 \neq 0$, or equivalently $x^2 \neq 4$, or equivalently $x \neq \pm 2$. So the domain is

$$\{x \mid x \neq \pm 2\}$$

or, written another way,

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

(b) \mathbb{R}

(c) $\{x \mid x \neq k\pi, k \in \mathbb{Z}\}$

(d) $(-\infty, 0]$

90. (a) 0

(b) $e^s - es^{e-1}$

(c) $(x^4 + 4x^3 - 2x - 2)e^x$

(d) $y' = \frac{d}{dx}(e^{bx^4}) = e^{bx^4} \frac{d}{dx}(bx^4) = e^{bx^4} \cdot 4bx^3 = 4bx^3 e^{bx^4}$.

(e) *Worked Solution:*

$$\begin{aligned}
 y' &= \frac{d}{dt}(e^{2t} \sin(4t)) \\
 &= \frac{d}{dt}(e^{2t}) \sin(4t) + e^{2t} \frac{d}{dt} \sin(4t) \\
 &= 2e^{2t} \sin(4t) + e^{2t} \cdot 4 \cos(4t) \\
 &= e^{2t}(2 \sin(4t) + 4 \cos(4t))
 \end{aligned}$$

(f) $-e^{x-e^x}$

(g) $\frac{(ad - bc)e^{-x}}{(c + de^{-x})^2}$

(h) $\frac{-e^{2x}(1 + 2x)}{2\sqrt{1 - xe^{2x}}}$

(i) $4 \sin(e^{\cos^2 t}) \cos(e^{\cos^2 t}) e^{\cos^2 t} \sin t \cos t$

91. The line has equation $y = \frac{e}{1-e}(x-1)$.92. (a) The domain of f is $[0, \infty)$ and $f^{-1}(x) = -\ln(1-x^2)$.

(b) *Worked Solution:* The domain of \ln is $(0, \infty)$. So the domain of $f(x) = \ln(3 - \ln x)$ is the set of x such that $x > 0$ and $3 - \ln x > 0$, or in other words $x > 0$ and $\ln x < 3$, or in other words $x > 0$ and $x < e^3$, or in other words the set $(0, e^3)$.

To find an expression for f^{-1} we first set $y = f(x)$, i.e. $y = \ln(3 - \ln x)$. Taking exponentials of both sides, we find $e^y = 3 - \ln x$, so that $\ln x = 3 - e^y$, so that $x = e^{3-e^y}$. Next we interchange x and y to find $y = e^{3-e^x}$, so that

$$f^{-1}(x) = e^{3-e^x}.$$

(c) The domain is $(0, \infty)$, and $f^{-1}(x) = \ln(e^x + 1)$.93. (a) $2 + \ln x$

(b) $-\sin(\ln x)/x$

(c) *Worked Solution:* $f'(x) = \frac{d}{dx}(\ln(\cos x)) = \frac{1}{\cos x} \frac{d}{dx}(\cos x) = \frac{1}{\cos x}(-\sin x) = -\tan x$.

(d) $\frac{1}{4x(\ln x)^{3/4}}$

(e) *Worked Solution:* Since $\frac{d}{dx} \log_{10}(x) = \frac{1}{x \ln 10}$, the chain rule gives us

$$\frac{d}{dx} \log_{10}(x^2 + 1) = \frac{1}{(x^2 + 1) \ln 10} \cdot \frac{d}{dx}(x^2 + 1) = \frac{1}{(x^2 + 1) \ln 10} \cdot 2x = \frac{2x}{(x^2 + 1) \ln 10}.$$

(f) $4x^3 + \ln(4) \cdot 4^x$

(g) $\frac{-1}{x(x^2-1)}$

(h) $\frac{-1}{\sqrt{x^2-1}}$

(i) $\frac{x}{x-1}$

95. (a) One-to-one.

(b) *Worked Solution:* The function is not one-to-one because $f(0) = 1 = f(4)$ but $0 \neq 4$.

(c) Not one-to-one.

(d) *Worked Solution:* The function is one-to-one because if $g(x_1) = g(x_2)$ then $\sqrt[4]{x_1} = \sqrt[4]{x_2}$, and by taking fourth powers we find that $x_1 = x_2$.

(e) Not one-to-one.

(f) *Worked Solution:* The function is one-to-one because no horizontal line crosses its graph more than once.

96. (a) $f^{-1}(3) = 1$ and $f^{-1}(-1) = -1$.

(b) $h^{-1}(10) = 8$ and $h^{-1}(2) = 1$.

98. (a) $f^{-1}(x) = \frac{4-x}{3}$

(b) $f^{-1}(x) = \frac{x^2-2x}{2}$

(c) $g^{-1}(x) = 1 + \sqrt{1+x}$

(d) *Worked Solution:* First we solve the equation $h(x) = y$ to express x in terms of y . If $\frac{1-2\sqrt{x}}{1+2\sqrt{x}} = y$ then $y + 2y\sqrt{x} = 1 - 2\sqrt{x}$, so that $\sqrt{x}(2y+2) = 1-y$ and consequently $\sqrt{x} = \frac{1-y}{2y+2}$ so that $x = \left(\frac{1-y}{2y+2}\right)^2$. Next we replace x with y and *vice versa* to obtain $y = \left(\frac{1-x}{2x+2}\right)^2$. Finally, this equation is $y = h^{-1}(x)$, so that

$$h^{-1}(x) = \left(\frac{1-x}{2x+2}\right)^2.$$

(e) *Worked Solution:* First we solve the equation $k(t) = y$ to express t in terms of y . If $y = 4t^2 - 2t$ then $4t^2 - 2t - y = 0$, so by the quadratic formula $t = \frac{2 \pm \sqrt{4+16y}}{8} = \frac{2 \pm 2\sqrt{1+4y}}{8} = \frac{1}{4} \pm \frac{1}{4}\sqrt{1+4y}$. Now since $t \geq 1/4$, we must have $t = \frac{1}{4} + \frac{1}{4}\sqrt{1+4y}$.

Next, we interchange y and t to obtain $y = \frac{1}{4} + \frac{1}{4}\sqrt{1+4t}$. Finally, this is the equation $y = k^{-1}(t)$, so that

$$k^{-1}(t) = \frac{1}{4} + \frac{1}{4}\sqrt{1+4t}.$$

(f) $f^{-1}(x) = \frac{1+x}{2-4x}$.

99. (a) *Worked Solution:* We know that $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$. Since $f(0) = 3$, we have $f^{-1}(3) = 0$, so that $(f^{-1})'(3) = \frac{1}{f'(0)}$. Now $f'(x) = 9x^2 + 4x + 8$, so that $f'(0) = 8$, and $(f^{-1})'(3) = 1/8$.

(b) $(f^{-1})'(3) = 1/2$

(c) $(f^{-1})'(4) = 1/2$

(d) $(f^{-1})'(1) = 12/13$

100. Using the quotient rule gives us

$$G'(x) = \frac{f^{-1}(x) \cdot \frac{d}{dx}(f(x)^2) - f(x)^2(f^{-1})'(x)}{f^{-1}(x)^2} = \frac{2f(x)f'(x)f^{-1}(x) - f(x)^2/f'(f^{-1}(x))}{f^{-1}(x)^2}$$

and in particular

$$G'(2) = \frac{2f(2)f'(2)f^{-1}(2) - f(2)^2/f'(f^{-1}(2))}{f^{-1}(2)^2}.$$

Now, since $f(3) = 2$, we have $f^{-1}(2) = 3$, and so

$$G'(2) = \frac{6f(2)f'(2) - f(2)^2/f'(3)}{3^2} = \frac{6 \cdot 1 \cdot 3 - 1^2/4}{3^2} = 71/36.$$

101. (a): Expanding gives

$$\begin{aligned} \int (2t-7)^3 dt &= \int (2t)^3 + 3 \cdot (-7)(2t)^2 + 3 \cdot (-7)^2(2t) + (-7)^3 dt \\ &= \int 8t^3 - 84t^2 + 294t - 343 dt \\ &= 8\frac{t^4}{4} - 84\frac{t^3}{3} + 294\frac{t^2}{2} - 343t + C \\ &= 2t^4 - 28t^3 + 147t^2 - 343t + C. \end{aligned}$$

(b): Use

$$u = 2t - 7, \quad du = 2dt$$

to get

$$\begin{aligned} \int (2t-7)^3 dt &= \int u^3 \frac{dt}{2} \\ &= \frac{u^4}{8} \\ &= \frac{(2t-7)^4}{8} + C. \end{aligned}$$

Note that the two answers differ by a constant.

102. (a): Using the angle-sum trig identities, we get

$$\begin{aligned}
 \sin(x)^3 &= \sin(x) \sin(x)^2 \\
 &= \sin(x) \frac{1 - \cos(2x)}{2} \\
 &= \frac{1}{2} (\sin(x) - \sin(x) \cos(2x)) \\
 &= \frac{1}{2} (\sin(x) - \frac{1}{2} (\sin(x + 2x) + \sin(x - 2x))) \\
 &= \frac{1}{2} \sin(x) - \frac{1}{4} \sin(3x) + \frac{1}{4} \sin(x) \\
 &= \frac{3}{4} \sin(x) - \frac{1}{4} \sin(3x),
 \end{aligned}$$

and therefore

$$\begin{aligned}
 \int \sin(x)^3 dx &= \int \left(\frac{3}{4} \sin(x) - \frac{1}{4} \sin(3x) \right) dx \\
 &= -\frac{3}{4} \cos(x) + \frac{1}{12} \cos(3x) + C.
 \end{aligned}$$

(b): Use

$$u = \cos(x), \quad du = -\sin(x) dx$$

to get

$$\begin{aligned}
 \int \sin(x)^3 dx &= \int (1 - \cos(x)^2) \sin(x) dx \\
 &= -\int (1 - u^2) du \\
 &= -u + \frac{u^3}{3} + C \\
 &= -\cos(x) + \frac{\cos(x)^3}{3} + C.
 \end{aligned}$$

These answers might look very different; however, they can be shown to be equal using more trig double-angle identities.

103. (a): Using

$$u = \cos(t), \quad du = -\sin(t) dt,$$

we get

$$\begin{aligned}
 \int \sin(t) (\cos(t)^4 + 6 \cos(t)^3 - 6) dt &= -\int (u^4 + 6u^3 - 6) du \\
 &= -\frac{u^5}{5} + \frac{6u^4}{4} + 6u + C \\
 &= -\frac{\cos(t)^5}{5} + \frac{3 \cos(t)^4}{2} + 6 \cos(t) + C.
 \end{aligned}$$

(b): Using

$$u = \ln(x), \quad du = \frac{1}{x} dx,$$

we get

$$\begin{aligned} \int \frac{1}{x} (\ln(x) - \ln(x)^3 + 2 \cos(\ln(x))) dx &= \int u - u^3 + 2 \cos(u) du \\ &= \frac{u^2}{2} - \frac{u^4}{4} + 2 \sin(u) \\ &= \frac{\ln(x)^2}{2} - \frac{\ln(x)^4}{4} + 2 \sin(\ln(x)). \end{aligned}$$

(c): Note that $w^w = e^{w \ln(w)}$. Using

$$u = w \ln(w), \quad du = w \frac{1}{w} + \ln(w) dw = 1 + \ln(w) dw,$$

we get

$$\begin{aligned} \int w^w (1 + \ln(w)) dw &= \int e^u du \\ &= e^u + C \\ &= e^{w \ln(w)} + C \\ &= w^w + C. \end{aligned}$$

It is also possible to do this one by substituting $u = w^w$.

(d): Using

$$u = 3x^2 + 5, \quad du = 6x dx,$$

we get

$$\int \frac{x \cos(3x^2 + 5)}{\sin(3x^2 + 5) + 6} dx = \int \frac{\cos(u)}{6(\sin(u) + 6)} du.$$

We need a further substitution. Use

$$v = \sin(u) + 6, \quad dv = \cos(u) du$$

to get

$$\begin{aligned} \int \frac{x \cos(3x^2 + 5)}{\sin(3x^2 + 5) + 6} dt &= \int \frac{\cos(u)}{6(\sin(u) + 6)} du \\ &= \int \frac{1}{6v} dv \\ &= \frac{\ln |v|}{6} + C \\ &= \frac{\ln |(\sin(u) + 6)|}{6} + C \\ &= \frac{\ln |(\sin(3x^2 + 5) + 6)|}{6} + C. \end{aligned}$$

One could do this with one substitution, namely $u = \sin(3x^2 + 5) + 6$.

(e): Using

$$u = \sqrt{\tan(x) + 17}, \quad du = \frac{1}{2} \sec(x)^2 (\tan(x) + 17)^{-1/2} dx,$$

we get

$$\begin{aligned} \int \sec(x)^2 \frac{e^{\sqrt{\tan(x)+17}}}{\sqrt{\tan(x)+17}} dx \\ &= \int e^u 2du \\ &= 2e^u + C \\ &= 2e^{\sqrt{\tan(x)+17}} + C. \end{aligned}$$

104. (a): Note that

$$\frac{A}{B+Cx} = A \frac{1}{B+Cx}.$$

It therefore seems that we should substitute

$$u = B + Cx, \quad du = C, dx,$$

which gives

$$\begin{aligned} \int \frac{A}{B+Cx} dx &= A \int \frac{1}{u} \frac{1}{C} du \\ &= \frac{A}{C} \ln |u| + K \\ &= \frac{A}{C} \ln |B+Cx| + K. \end{aligned}$$

(b): Note that

$$\frac{A}{B+Cx^2} = \frac{A}{B} \frac{1}{1+(C/B)x^2} = \frac{A}{B} \frac{1}{1+(\sqrt{(C/B)}x)^2}.$$

Therefore it makes sense to substitute

$$u = \sqrt{\frac{C}{B}}x, \quad du = \sqrt{\frac{C}{B}} dx,$$

which gives

$$\begin{aligned} \int \frac{A}{B+Cx^2} dx &= \frac{A}{B} \int \frac{1}{1+u^2} \sqrt{\frac{B}{C}} du \\ &= \frac{A\sqrt{B}}{B\sqrt{C}} \tan^{-1}(u) + K \\ &= \frac{A}{\sqrt{BC}} \tan^{-1} \left(\sqrt{\frac{C}{B}}x \right) + K. \end{aligned}$$

(c) Note that

$$\frac{A}{\sqrt{B - Cx^2}} = \frac{A}{\sqrt{B}} \frac{1}{\sqrt{1 - (\sqrt{C/B}x)^2}},$$

and so we substitute

$$u = \sqrt{\frac{C}{B}}x, \quad du = \sqrt{\frac{C}{B}} dx.$$

This leads to

$$\begin{aligned} \int \frac{A}{\sqrt{B - Cx^2}} dx &= \frac{A}{\sqrt{B}} \int \frac{1}{\sqrt{1 - u^2}} \sqrt{\frac{B}{C}} du \\ &= \frac{A}{\sqrt{C}} \sin^{-1}(u) + K \\ &= \frac{A}{\sqrt{C}} \sin^{-1}\left(\sqrt{\frac{C}{B}}x\right) + K. \end{aligned}$$

105. (a): Since $\rho'(x) = e^{-x^2}$, we have (by definition)

$$\int \frac{2e^{-x^2}}{\sqrt{\pi}} dx = \frac{2\rho(x)}{\sqrt{\pi}} + C.$$

(b): Note that

$$xe^{-1+2x^2-x^4} = xe^{-(1-x^2)^2},$$

which suggests the substitution

$$u = 1 - x^2, \quad du = -2x dx.$$

With this, we get

$$\begin{aligned} \int xe^{-1+2x^2-x^4} dx &= -\frac{1}{2} \int e^{-u^2} du \\ &= -\frac{1}{2}\rho(u) \\ &= -\frac{1}{2}\rho(1 - x^2). \end{aligned}$$

(c): First use the substitution

$$u = t^3, \quad du = 3t^2 dt$$

to get

$$\int t^2 \rho(t^3) e^{-t^6} dt = \frac{1}{3} \int \rho(u) e^{-u^2} du.$$

Now use the further substitution

$$v = \rho(u), \quad dv = e^{-u^2} du$$

to get

$$\begin{aligned}\int t^2 \rho(t^3) e^{-t^6} dt &= \frac{1}{3} \int \rho(u) e^{-u^2} du \\ &= \frac{1}{3} \int v dv \\ &= \frac{1}{6} v^2 + C \\ &= \frac{1}{6} \rho(u)^2 + C \\ &= \frac{1}{6} \rho(t^3)^2 + C.\end{aligned}$$

106. (a): This area is given by the integral

$$\begin{aligned}\int_0^\pi \sin(x) dx &= -\cos(x) \Big|_{x=0}^\pi \\ &= -\cos(\pi) - \cos(0) = 1 - 1 = 2.\end{aligned}$$

(b): The two curves intersect when

$$x^3 - 25x = -x^3 - x + 32$$

which can be rearranged as

$$\begin{aligned}0 &= 2x^3 - 24x - 32 \\ &= 2(x - 4)(x + 2)^2.\end{aligned}$$

Thus, the points of intersection are

$$(-2, (-2)^3 - 25(-2)) = (-2, 42)$$

and

$$(4, 4^3 - 25 \cdot 4) = (4, -36).$$

Note (e.g., by evaluating at 0) that the curve $y = x^3 - 25x$ lies below the curve $y = -x^3 - x - 32$ for $x \in (-2, 4)$.

As in Example 1.5.6, this area is given by

$$\begin{aligned}\int_{-2}^4 (-x^3 - x + 32) - (x^3 - 25x) dx &= \int_{-2}^4 -2x^3 + 24x + 32 dx \\ &= -\frac{2x^4}{4} + \frac{24x^2}{2} + 32x \Big|_{x=-2}^4 \\ &= -\frac{4^4}{2} + 12(4) + 32(4) - \left(-\frac{(-2)^4}{2} + 12(-2)^2 + 32(-2) \right) \\ &= 216.\end{aligned}$$

(c): As in Example 1.5.7 (b), we integrate over y instead of x , since it is very difficult to express the first curve in the form $y = f(x)$. The area is then given by

$$\int_{\ln(\pi)}^{\ln(2\pi)} -e^y \sin(e^y) - (-y) dy = \int_{\ln(\pi)}^{\ln(2\pi)} -e^y \sin(e^y) dy + \int_{\ln(\pi)}^{\ln(2\pi)} y dy$$

For the first of these integrals, we want to substitute

$$u = e^y, \quad du = e^y dy$$

to get

$$\begin{aligned} \int e^y \sin(e^y) dy &= \int \sin(u) du \\ &= -\cos(u) + C \\ &= -\cos(e^y) + C. \end{aligned}$$

Hence, the area is

$$\begin{aligned} \int_{\ln(\pi)}^{\ln(2\pi)} e^y \sin(e^y) - (-y) dy &= \cos(e^y) + \frac{y^2}{2} \Big|_{y=\ln(\pi)}^{\ln(2\pi)} \\ &= \cos(2\pi) + \frac{\ln(2\pi)^2}{2} - \left(\cos(\pi) + \frac{\ln(\pi)^2}{2} \right) \\ &= 1 + \frac{(\ln(2) + \ln(\pi))^2}{2} - (-1) - \frac{\ln(\pi)^2}{2} \\ &= 2 + \frac{\ln(2)^2}{2} + \ln(2) \ln(\pi) + \frac{\ln(\pi)^2}{2} - \frac{\ln(\pi)^2}{2} \\ &= 2 + \frac{\ln(2)^2}{2} + \ln(2) \ln(\pi). \end{aligned}$$

(d): It is very difficult to write the first curve in the form $y = f(x)$ or (equally) the second curve in the form $x = f(y)$. So, this question requires a slightly different approach than what we did before.

We break this region into two pieces along the line $y = x$. We can see that each piece is a reflection of the other, so the total area is

$$2 \int_0^1 \frac{x^2 \sqrt{10 - x^3}}{3} dx.$$

To solve this integral, we use the substitution

$$u = 10 - x^3, \quad du = -3x^2 dx.$$

This gives

$$\begin{aligned} \int \frac{x^2 \sqrt{10 - x^3}}{3} dx &= -\frac{1}{9} \int \sqrt{u} du \\ &= -\frac{1}{9} \frac{u^{3/2}}{3/2} + C \\ &= -\frac{2}{27} (10 - x^3)^{3/2} + C. \end{aligned}$$

and therefore, the area is

$$\begin{aligned} 2 \int_0^1 \frac{x^2 \sqrt{10-x^3}}{3} dx &= -\frac{4}{27} (10-x^3)^{3/2} \Big|_{x=0}^1 \\ &= -\frac{4}{27} (9^{3/2} - 10^{3/2}) \\ &= -4 + \frac{40\sqrt{10}}{27}. \end{aligned}$$

107. (a): By FTC1 and the Chain Rule (as in Example 1.5.3 (ii)), we have

$$f'(x) = e^{-(2x)^2} \cdot \left(\frac{d}{dx} 2x \right) = 2e^{-4x^2}.$$

(b): To do this, first rewrite it as a difference of integrals:

$$\begin{aligned} g(x) &= \int_{-x}^0 \sqrt[3]{\cos(t)} dt + \int_0^{x^2} \sqrt[3]{\cos(t)} dt \\ &= - \int_0^{-x} \sqrt[3]{\cos(t)} dt + \int_0^{x^2} \sqrt[3]{\cos(t)} dt. \end{aligned}$$

Now we differentiate each term using FTC1 and the Chain Rule, as before.

$$g'(x) = \sqrt[3]{\cos(-x)} + 2x \sqrt[3]{\cos(x^2)}.$$

(c): This one is a bit different, because the integrand contains x . However, we can factor x out of the integrand, then use the Product Rule.

$$\begin{aligned} h(x) &= \int_x^0 -x^2 \sqrt{1+t^3} dt \\ &= x^2 \int_0^x \sqrt{1+t^3} dt \\ h'(x) &= 2x \int_0^x \sqrt{1+t^3} dt + x^2 \sqrt{1+t^3}. \end{aligned}$$

It is alright to leave the answer like this (with a definite integral).

108. (a): Using the answer to Question 4(b) on Exercise Sheet 1, we have

$$\begin{aligned} \int_0^{5/2} \frac{1}{4x^2+25} dx &= \frac{1}{10} \tan^{-1} \left(\frac{2x}{5} \right) \Big|_{x=0}^{5/2} \\ &= \frac{1}{10} (\tan^{-1}(1) - \tan^{-1}(0)) \\ &= \frac{1}{10} \left(\frac{\pi}{4} - 0 \right) \\ &= \frac{\pi}{40}. \end{aligned}$$

(b): Use the substitution

$$u = 4x^2 + 25, \quad du = 8x \, dx$$

to get

$$\begin{aligned} \int \frac{x}{4x^2 + 25} \, dx &= \int \frac{1}{8u} \, du \\ &= \frac{\ln(u)}{8} + C \\ &= \frac{\ln(4x^2 + 25)}{8} + C. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^{5/2} \frac{x}{4x^2 + 25} \, dx &= \left. \frac{\ln(4x^2 + 25)}{8} \right|_{x=0}^{5/2} \\ &= \frac{1}{8} (\ln(25 + 25) - \ln(25)) \\ &= \frac{\ln(25) + \ln(2) - \ln(25)}{8} \\ &= \frac{\ln(2)}{8}. \end{aligned}$$

(c): Using the substitution

$$u = x^3 - 2x^{3/2} + 2, \quad du = 3x^2 - 2 \cdot \frac{3}{2} \cdot \sqrt{x} \, dx,$$

we get

$$\begin{aligned} \int \frac{x^2 - \sqrt{x}}{(x^3 - 2x^{3/2} + 2)^2} \, dx &= \frac{1}{3} \int \frac{1}{u^2} \, du \\ &= -\frac{1}{3u} + C \\ &= -\frac{1}{3(x^3 - 2x^{3/2} + 2)} + C. \end{aligned}$$

Hence,

$$\begin{aligned} \int_1^4 \frac{x^2 - \sqrt{x}}{(x^3 - 2x^{3/2} + 2)^2} \, dx &= \left. -\frac{1}{3(x^3 - 2x^{3/2} + 2)} \right|_{x=1}^4 \\ &= -\frac{1}{3} \left(-\frac{1}{64 - 16 + 2} - \frac{1}{1 - 2 + 2} \right) \\ &= -\frac{1}{3} \cdot \frac{1 - 50}{50} \\ &= \frac{49}{150}. \end{aligned}$$

(d): First, use the substitution

$$u = 1 - \cos(x)^2, \quad du = 2 \cos(x) \sin(x) dx$$

to get

$$\begin{aligned} & \int \sin(x) \cdot \cos(x) \cdot \cos\left(\frac{\pi}{e-1} \left(e^{1-\cos(x)^2} - 1\right)\right) \cdot e^{1-\cos(x)^2} dx \\ &= \int \cos\left(\frac{\pi}{e-1} (e^u - 1)\right) \cdot e^u du. \end{aligned}$$

Next substitute

$$v = \frac{\pi(e^u - 1)}{e - 1}, \quad dv = \frac{\pi(e^u)}{e - 1} du$$

to get

$$\begin{aligned} & \int \sin(x) \cdot \cos(x) \cdot \cos\left(\frac{\pi}{e-1} \left(e^{1-\cos(x)^2} - 1\right)\right) \cdot e^{1-\cos(x)^2} dx \\ &= \int \cos\left(\frac{\pi}{e-1} (e^u - 1)\right) \cdot e^u du \\ &= \frac{e-1}{\pi} \int \cos(v) dv \\ &= \frac{e-1}{\pi} \sin(v) \\ &= \frac{e-1}{\pi} \sin\left(\frac{\pi(e^u - 1)}{e-1}\right) \\ &= \frac{e-1}{\pi} \sin\left(\frac{\pi(e^{1-\cos(x)^2} - 1)}{e-1}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} & \int_0^{\pi/2} \sin(x) \cdot \cos(x) \cdot \cos\left(\frac{\pi}{e-1} \left(1 - e^{\cos(x)^2} - 1\right)\right) \cdot e^{1-\cos(x)^2} dx \\ &= \frac{e-1}{\pi} \sin\left(\frac{\pi(e^{1-\cos(x)^2} - 1)}{e-1}\right) \Bigg|_{x=0}^{\pi/2} \\ &= \frac{e-1}{\pi} \left(\sin\left(\frac{\pi(e^{1-\cos(\pi/2)^2} - 1)}{e-1}\right) - \sin\left(\frac{\pi(e^{1-\cos(0)^2} - 1)}{e-1}\right) \right) \\ &= \frac{e-1}{\pi} \left(\sin\left(\frac{\pi(e-1)}{e-1}\right) - \sin\left(\frac{\pi(1-1)}{e-1}\right) \right) \\ &= \frac{e-1}{\pi} (-1 - 1) \\ &= \frac{2(1-e)}{\pi}. \end{aligned}$$

109. (a): Use

$$\begin{aligned}u &= x^2, & dv &= \cos(2\pi x) dx, \\du &= 2x dx, & v &= \frac{\sin(2\pi x)}{2\pi}\end{aligned}$$

to get

$$\int x^2 \cos(2\pi x) dx = \frac{x^2 \sin(2\pi x)}{2\pi} - \frac{1}{\pi} \int x \sin(2\pi x) dx.$$

To solve the new integral, use Integration by Parts again, with

$$\begin{aligned}u &= x, & dv &= \sin(2\pi x) dx, \\du &= dx, & v &= -\frac{\cos(2\pi x)}{2\pi}.\end{aligned}$$

This gives

$$\begin{aligned}\int x^2 \cos(2\pi x) dx &= \frac{x^2 \sin(2\pi x)}{2\pi} - \frac{1}{\pi} \int x \sin(2\pi x) dx \\&= \frac{x^2 \sin(2\pi x)}{2\pi} - \frac{1}{\pi} \left(-\frac{x \cos(2\pi x)}{2\pi} + \frac{1}{2\pi} \int \cos(2\pi x) dx \right) \\&= \frac{x^2 \sin(2\pi x)}{2\pi} + \frac{1}{2\pi^2} \left(x \cos(2\pi x) - \frac{\sin(2\pi x)}{2\pi} \right) + C \\&= \frac{x^2 \sin(2\pi x)}{2\pi} + \frac{x \cos(2\pi x)}{2\pi^2} - \frac{\sin(2\pi x)}{4\pi^3} + C.\end{aligned}$$

(b): Using

$$\begin{aligned}u &= x, & dv &= \sqrt{2x+1} dx, \\du &= dx, & v &= \frac{2}{3} \cdot \frac{1}{2} (2x+1)^{3/2}\end{aligned}$$

we get

$$\begin{aligned}\int x\sqrt{2x+1} dx &= \frac{2}{3}x(2x+1)^{3/2} - \frac{1}{3} \int (2x+1)^{3/2} \\&= \frac{2}{3}x(2x+1)^{3/2} - \frac{1}{3} \cdot \frac{5}{2} (2x+1)^{5/2} + C \\&= \frac{2}{3}x(2x+1)^{3/2} - \frac{5}{6} (2x+1)^{5/2} + C.\end{aligned}$$

(c): Note that by a substitution ($w = e^x$), we have

$$\int e^x \cos(e^x) dx = \sin(e^x) + C.$$

So, we may use

$$\begin{aligned}u &= e^x, & dv &= e^x \cos(e^x) dx \\du &= e^x dx, & v &= \sin(e^x)\end{aligned}$$

and get

$$\begin{aligned}\int e^{2x} \cos(e^x) dx &= e^x \cos(e^x) - \int e^x \sin(e^x) dx \\ &= e^x \cos(e^x) + \cos(e^x) + C'.\end{aligned}$$

(d): We use

$$\begin{aligned}u &= \ln(x)^2, & dv &= x dx \\ du &= \frac{2 \ln(x)}{x} dx, & v &= \frac{x^2}{2}.\end{aligned}$$

This yields

$$\begin{aligned}\int x \ln(x)^2 dx &= \frac{x^2 \ln(x)^2}{2} - \int \frac{x^2 \ln(x)}{x} dx \\ &= \frac{x^2 \ln(x)^2}{2} - \int x \ln(x) dx.\end{aligned}$$

For the second integral use

$$\begin{aligned}u &= \ln(x), & dv &= x dx \\ du &= \frac{1}{x} dx, & v &= \frac{x^2}{2}.\end{aligned}$$

Then we get

$$\begin{aligned}\int x \ln(x)^2 dx &= \frac{x^2 \ln(x)^2}{2} - \int x \ln(x) dx \\ &= \frac{x^2 \ln(x)^2}{2} - \left(\frac{x^2 \ln(x)}{2} - \int \frac{x^2}{2x} dx \right) \\ &= \frac{x^2 \ln(x)^2}{2} - \frac{x^2 \ln(x)}{2} + \frac{x^2}{4} + C.\end{aligned}$$

(e): Here we use

$$\begin{aligned}u &= \ln(x)^2, & dv &= \ln(x) dx \\ du &= \frac{2 \ln(x)}{x} dx, & v &= x \ln(x) - x,\end{aligned}$$

and this gives

$$\begin{aligned}\int \ln(x)^3 dx &= \ln(x)^2(x \ln(x) - x) - 2 \int \frac{(x \ln(x) - x) \ln(x)}{x} dx \\ &= x \ln(x)^3 - x \ln(x)^2 - 2 \int \ln(x)^2 - \ln(x) dx \\ &= x \ln(x)^3 - x \ln(x)^2 \\ &\quad - 2(x \ln(x)^2 - 2x \ln(x) + 2x - (x \ln(x) - x)) + C \\ &= x \ln(x)^3 - 3x \ln(x)^2 + 6x \ln(x) - 6x + C.\end{aligned}$$

110. (a): We rewrite the integrand as a function of $\sin(t)$ followed by a single factor of $\cos(t)$, so that we can substitute $u = \sin(t)$:

$$\begin{aligned}\int \sin(t)^3 \cos(t)^3 dt &= \int \sin(t)^3 (1 - \sin(t)^2) \cos(t) dt \\ &= \int u^3 (1 - u^2) du \\ &= \frac{u^4}{4} - \frac{u^6}{6} + C \\ &= \frac{\sin(t)^4}{4} - \frac{\sin(t)^6}{6} + C.\end{aligned}$$

- (b): Use angle-sum identities to reduce the integrand:

$$\begin{aligned}\sin(x) \cos(2x)^2 &= \frac{1}{2} \sin(x) (1 + \cos(4x)) \\ &= \frac{1}{2} \sin(x) + \frac{1}{4} (\sin(5x) - \sin(3x)).\end{aligned}$$

Therefore,

$$\begin{aligned}\int \sin(x) \cos(2x)^2 dx &= \frac{1}{2} \int \sin(x) dx + \frac{1}{4} \int \sin(5x) - \sin(3x) dx \\ &= -\frac{1}{2} \cos(x) - \frac{1}{4 \cdot 5} \cos(5x) + \frac{1}{4 \cdot 3} \cos(3x) + C.\end{aligned}$$

- (c): Using angle-sum identities to reduce the integrand, we get

$$\begin{aligned}\int \sin(x) \sin(x+2) dx &= \frac{1}{2} \int \cos(2) - \cos(2x+2) dx \\ &= \frac{1}{2} \left(\cos(2) \cdot x - \frac{1}{2} \sin(2x+2) \right) + C.\end{aligned}$$

- (d): Substitute

$$u = \cos(x), \quad du = -\sin(x) dx$$

to get

$$\begin{aligned}\int \tan(x) \sec(x)^2 dx &= \int \frac{\sin(x)}{\cos(x)^3} dx \\ &= -\int \frac{1}{u^3} du \\ &= \frac{1}{2u^2} + C \\ &= \frac{1}{2} \sec(x)^2.\end{aligned}$$

(There are other ways to do this one.)

(e): Substitute

$$u = \tan(x), \quad du = \sec(x)^2 dx$$

to get

$$\begin{aligned} \int \tan(x)^2 \sec(x)^2 dx &= \int u^2 du \\ &= \frac{u^3}{3} + C \\ &= \frac{\tan(x)^3}{3} + C. \end{aligned}$$

(f): This one is very similar to Example 1.7.3 (v), and we use the same idea to solve it. Write the integrand as

$$\frac{1}{\sin(x)} = \csc(x) = \frac{\csc(x)^2 + \csc(x) \cot(x)}{\csc(x) + \cot(x)},$$

and then substitute

$$u = \csc(x) + \cot(x), \quad du = -\csc(x)^2 - \csc(x) \cot(x) dx$$

to get

$$\begin{aligned} \int \frac{1}{\sin(x)} dx &= \int \frac{\csc(x)^2 + \csc(x) \cot(x)}{\csc(x) + \cot(x)} dx \\ &= -\int \frac{1}{u} du \\ &= -\ln |u| + C \\ &= -\ln |\csc(x) + \cot(x)| + C. \end{aligned}$$

(g): There is no good substitution to do here, so instead we turn to IBP. Try

$$\begin{aligned} u &= \cot(x), & dv &= \cot(x) \csc(x) dx \\ du &= -\csc(x)^2 dx & v &= -\csc(x) \end{aligned}$$

to get

$$\begin{aligned} \int \cot(x)^2 \csc(x) dx &= -\cot(x) \csc(x) - \int \csc(x)^3 dx \\ &= -\cot(x) \csc(x) - \int \csc(x)(1 + \cot(x)^2) dx \\ &= -\cot(x) \csc(x) + \ln |\csc(x) + \cot(x)| - \int \csc(x) \cot(x)^2 dx, \end{aligned}$$

using (f). Now we solve for the integral we want:

$$\begin{aligned} 2 \int \cot(x)^2 \csc(x) dx &= -\cot(x) \csc(x) + \ln |\csc(x) + \cot(x)|, \\ \int \cot(x)^2 \csc(x) dx &= -\frac{1}{2} (\cot(x) \csc(x) - \ln |\csc(x) + \cot(x)|). \end{aligned}$$

111. (a): Use IBP with

$$\begin{aligned}u &= x, & dv &= \sec(x)^2 dx, \\ du &= dx, & v &= \tan(x).\end{aligned}$$

We get

$$\begin{aligned}\int x \sec(x)^2 dx &= x \tan(x) - \int \tan(x) dx \\ &= x \tan(x) - \ln |\cos(x)| + C,\end{aligned}$$

using Example 1.7.3 (iii).

(b): We will use IBP.

Note that, via the substitution $u = 1 + x^2$, we have

$$\int x \sqrt[3]{1+x^2} dx = \frac{1}{2} \int u^{1/3} du = \frac{3}{8} u^{4/3} + C = \frac{3}{8} (1+x^2)^{4/3} + C.$$

We use this to compute v in the following

$$\begin{aligned}u &= x^2, & dv &= x \sqrt[3]{1+x^2} dx \\ du &= 2x, & v &= \frac{3}{8} (1+x^2)^{4/3},\end{aligned}$$

so that

$$\begin{aligned}\int x^3 \sqrt[3]{1+x^2} dx &= \frac{3}{8} \left(x^2 (1+x^2)^{4/3} - \int 2x (1+x^2)^{4/3} dx \right) \quad (\text{IBP}) \\ &= \frac{3}{8} \left(x^2 (1+x^2)^{4/3} - \int u^{4/3} dx \right) \quad (\text{subst. } u = 1+x^2) \\ &= \frac{3}{8} \left(x^2 (1+x^2)^{4/3} - \frac{3}{7} u^{7/3} \right) + C \\ &= \frac{3}{8} x^2 (1+x^2)^{4/3} - \frac{9}{56} (1+x^2)^{7/3} + C.\end{aligned}$$

(c): Use IBP with

$$u = x, \quad dv = \cos(x)e^x dx \quad du = dx, \quad v = \frac{e^x(\sin(x) + \cos(x))}{2}$$

(where we used Example 1.6.9 to compute v). This leads to

$$\begin{aligned}\int x \cos(x)e^x dx &= \frac{xe^x(\sin(x) + \cos(x))}{2} - \frac{1}{2} \int e^x(\sin(x) + \cos(x)) dx \\ &= \frac{xe^x(\sin(x) + \cos(x))}{2} - \frac{1}{2} \left(\frac{e^x(\sin(x) - \cos(x))}{2} + \frac{e^x(\sin(x) + \cos(x))}{2} \right) + C \\ &= \frac{e^x(x \sin(x) + x \cos(x) - \sin(x))}{2} + C\end{aligned}$$

(using Example 1.6.9 again.)

(d): Use the substitution $w = \sqrt[3]{3x+7}$, or in other words,

$$x = \frac{w^3 - 7}{3}, \quad dx = w^2 dw$$

to get

$$\int e^{\sqrt[3]{3x+7}} dx = \int w^2 e^w dw.$$

Now use IBP with

$$\begin{aligned} u &= w^2, & dv &= e^w dw, \\ du &= 2w dw, & v &= e^w, \end{aligned}$$

which produces

$$\int w^2 e^w dw = w^2 e^w - 2 \int w e^w dw.$$

Use IBP again with

$$\begin{aligned} u &= w, & dv &= e^w dw, \\ du &= dw, & v &= e^w, \end{aligned}$$

to get

$$\begin{aligned} \int w^2 e^w dw &= w^2 e^w - 2 \int w e^w dw \\ &= w^2 e^w - 2w e^w + 2 \int e^w dw \\ &= w^2 e^w - 2w e^w + 2e^w + C. \end{aligned}$$

Putting that back into our original integral we have

$$\begin{aligned} \int e^{\sqrt[3]{3x+7}} dx &= \int w^2 e^w dw \\ &= w^2 e^w - 2w e^w + 2e^w + C \\ &= (3x+7)^{2/3} e^{\sqrt[3]{3x+7}} - 2\sqrt[3]{3x+7} e^{\sqrt[3]{3x+7}} + 2e^{\sqrt[3]{3x+7}} + C. \end{aligned}$$

(e): Use IBP with

$$\begin{aligned} u &= \ln((\sin(x))^2), & dv &= \sin(x) dx. \\ du &= \frac{1}{\sin(x)^2} \cdot 2 \sin(x)(x) \cos(x), & v &= -\cos(x). \end{aligned}$$

This produces

$$\begin{aligned}
 \int \sin(x) \ln((\sin(x)^2)) dx &= -\ln((\sin(x)^2)) \cos(x) - 2 \int \frac{-\cos(x) \sin(x) \cos(x)}{\sin(x)^2} dx \\
 &= -\ln((\sin(x)^2)) \cos(x) + 2 \int \frac{\cos(x)^2}{\sin(x)} dx \\
 &= -\ln((\sin(x)^2)) \cos(x) + 2 \int \frac{1 - \sin(x)^2}{\sin(x)} dx \\
 &= -\ln((\sin(x)^2)) \cos(x) + 2 \int \csc(x) - \sin(x) dx \\
 &= -\ln((\sin(x)^2)) \cos(x) - 2 \ln |\csc(x) + \cot(x)| + 2 \cos(x) + C,
 \end{aligned}$$

using question 1(f) at the end.

112. By induction. The base case is $n = 0$, where the equation reads

$$\int 1 dx = 0! x \frac{(-1)^{0-0}}{0!} \ln(x)^0 + C.$$

The right-hand side simplifies to $x + C$, so we know this is correct.

For $n > 0$, we use IBP with

$$\begin{aligned}
 u &= \ln(x)^n, & dv &= 1 dx \\
 du &= \frac{n \ln(x)^{n-1}}{x}, & v &= x.
 \end{aligned}$$

This gives

$$\begin{aligned}
 \int \ln(x)^n dx &= x \ln(x)^n - n \int x \cdot \frac{\ln(x)^{n-1}}{x} dx \\
 &= x \ln(x)^n - n \int \ln(x)^{n-1} dx \\
 &= x \ln(x)^n - n(n-1)! x \sum_{k=0}^{n-1} \frac{(-1)^{n-1-k}}{k!} \ln(x)^k + C \quad \text{induction} \\
 &= \frac{n!(-1)^{n-n}}{n!} x \ln(x)^n + n! x \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{k!} \ln(x)^k + C \\
 &= n! x \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} \ln(x)^k + C.
 \end{aligned}$$

113. (a): Use the trig substitution

$$x = \sec(u), \quad dx = \sec(u) \tan(u) du,$$

and assume $\tan(u) \geq 0$ so that

$$\sqrt{x^2 - 1} = \tan(u).$$

We get

$$\begin{aligned}
 \int \frac{1}{(x^2 - 1)^{3/2}} dx &= \int \frac{\sec(u) \tan(u)}{\tan(u)^3} du \\
 &= \int \frac{\cos(u)}{\sin(u)^2} du \\
 &= \int \frac{1}{w^2} dw \quad (\text{subst. } w = \sin(u)) \\
 &= -\frac{1}{w} + C \\
 &= -\frac{1}{\sin(u)} + C \\
 &= -\frac{\sec(u)}{\tan(u)} + C \\
 &= -\frac{x}{\sqrt{x^2 - 1}} + C.
 \end{aligned}$$

(b): Completing the square, we have $-(x^2 + 4x + 3) = 1 - (x + 2)^2$. Use the trig substitution

$$x + 2 = \sin(u), \quad dx = \cos(u) du,$$

and assume $\cos(u) \geq 0$ so that

$$\sqrt{-(x^2 + 4x + 3)} = \cos(u).$$

This gives

$$\begin{aligned}
 \int x \sqrt{-(x^2 + 4x + 3)} dx &= \int (\sin(u) - 2) \cos(u) \cos(u) du \\
 &= \int \sin(u) \cos(u)^2 du - 2 \int \cos(u)^2 du
 \end{aligned}$$

For the first of these, by substituting

$$w = \cos(u), \quad dw = -\sin(u) du,$$

we get

$$\begin{aligned}
 \int \sin(u) \cos(u)^2 du &= -\int w^2 dw \\
 &= -\frac{w^3}{3} + C \\
 &= -\frac{\cos(u)^3}{3} + C.
 \end{aligned}$$

For the second integral, use a double-angle formula to get

$$\begin{aligned}
 \int \cos(u)^2 du &= \frac{1}{2} \int 1 + \cos(2u) du \\
 &= \frac{1}{2} \left(u + \frac{1}{2} \sin(2u) \right) + C' \\
 &= \frac{1}{2} (u + \sin(u) \cos(u)) + C'.
 \end{aligned}$$

Altogether, we get

$$\begin{aligned} \int x\sqrt{-(x^2+4x+3)} dx &= \int \sin(u) \cos(u)^2 du - 2 \int \cos(u)^2 du \\ &= -\frac{1}{3} \cos(u)^3 - u - \sin(u) \cos(u) + C'' \\ &= -\frac{1}{3} (-(x^2+4x+3))^{3/2} - \sin^{-1}(x+2) \\ &\quad - (x+2)\sqrt{-(x^2+4x+3)} + C'' \end{aligned}$$

(c): Completing the square gives

$$9t^2 - 6t + 5 = (3t - 1)^2 + 4 = 4 \left(\left(\frac{3t - 1}{2} \right)^2 + 1 \right).$$

We therefore want to substitute

$$3t - 1 = 2 \tan(u), \quad 3 dt = 2 \sec(u)^2 du,$$

and assume $\sec(u) \geq 0$ so that

$$\sqrt{9t^2 - 6t + 5} = 2 \sqrt{\left(\frac{3t - 1}{2} \right)^2 + 1} = 2 \sec(u).$$

We get

$$\begin{aligned} \int \frac{1}{(3t - 1)^2 \sqrt{9t^2 - 6t + 5}} dt &= \frac{1}{3} \int \frac{\sec(u)^2}{2 \tan(u)^2 \sec(u)} du \\ &= \frac{1}{6} \int \frac{\cos(u)}{\sin(u)^2} du \\ &= \frac{1}{6} \int \frac{1}{w^2} dw \quad (\text{subst. } w = \sin(u)) \\ &= -\frac{1}{6w} + C \\ &= -\frac{1}{6 \sin(u)} + C \\ &= -\frac{\sec(u)}{6 \tan(u)} + C \\ &= -\frac{\sqrt{9t^2 - 6t + 5}}{12(3t - 1)} + C. \end{aligned}$$

114. (a): The denominator factors as

$$x^2 - x - 12 = (x - 4)(x + 3).$$

Since the numerator has degree 1 greater than the denominator, the integrand can be represented as

$$\frac{x^3 + x^2 - 13x - 63}{x^2 - x - 12} = \frac{A}{x + 3} + B \frac{x - 4}{x + 3} + Cx + D.$$

Multiplying by $(x^2 - x - 12)$ gives

$$\begin{aligned} x^3 + x^2 - 13x - 63 &= A(x - 4) + B(x + 3) + (Cx + D)(x^2 - x - 12) \\ &= Cx^3 + (-C + D)x^2 + (A + B - 12C - D)x + (-4A + 3B - 12D) \end{aligned}$$

giving four equations,

$$\begin{aligned} (1) : \quad 1 &= C, \\ (2) : \quad 1 &= -C + D = -1 + D, \\ &D = 2, \\ (3) : \quad -13 &= A + B - 12C - D = A + B - 12 - 2, \\ (4) : \quad -63 &= -4A + 3B - 12D = -4A + 3B - 12(2). \end{aligned}$$

We solve this by

$$\begin{aligned} 4(3) + (4) : \quad -115 &= 7B - 80 \\ &B = -35/7 = -5, \\ (3) : \quad -13 &= A + (-5) - 12 - 2 \\ &A = 6. \end{aligned}$$

Using this solution we now solve the integral,

$$\begin{aligned} \int \frac{x^3 + x^2 - 13x - 63}{x^2 - x - 12} dx &= \int \frac{6}{x + 3} - \frac{5}{x - 4} + x + 2 dx \\ &= 6 \ln |x + 3| - 5 \ln |x - 4| + \frac{1}{2}x^2 + 2x + C. \end{aligned}$$

(b): We factor the denominator as

$$x^3 + 8x^2 + 5x - 14 = (x - 1)(x + 2)(x + 7).$$

Since the degree of the numerator is below that of the denominator, we must be able to express the integrand as

$$\frac{6x^2 + 49x - 31}{x^3 + 8x^2 + 5x - 14} = \frac{A}{x - 1} + \frac{B}{x + 2} + \frac{C}{x + 7}.$$

Multiply by $(x^3 + 8x^2 + 5x - 14)$ to get

$$6x^2 + 49x - 31 = A(x^2 + 9x + 14) + B(x^2 + 6x - 7) + C(x^2 + x - 2),$$

which becomes the system

$$\begin{aligned} (1) : \quad 6 &= A + B + C, \\ (2) : \quad 49 &= 9A + 6B + C, \\ (3) : \quad -31 &= 14A - 7B - 2C. \end{aligned}$$

Solve this:

$$\begin{aligned}
 (4) : (2) - (1) : \quad 43 &= 8A + 5B, \\
 (5) : (3) + 2(1) : \quad -19 &= 16A - 5B, \\
 (4) + (5) : \quad 24 &= 24A, \\
 &A = 1, \\
 (4) : \quad 43 &= 8 + 5B, \\
 &B = 35/5 = 7 \\
 (1) : \quad 6 &= 1 + 7 + C, \\
 &C = -2.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 \int \frac{6x^2 + 49x - 31}{x^3 + 8x^2 + 5x - 14} dx &= \int \frac{1}{x-1} + \frac{7}{x+2} - \frac{2}{x+7} dx \\
 &= \ln|x-1| + 7\ln|x+2| - 2\ln|x+7| + C.
 \end{aligned}$$

(c): The denominator factors as

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3.$$

Since the degree of the numerator matches that of the denominator, PFD will give

$$\frac{x^3 - 2x^2 - 9x + 16}{x^3 - 6x^2 + 12x - 8} = A + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}.$$

Multiplying by $(x-2)^3$ yields

$$x^3 - 2x^2 - 9x + 16 = A(x^3 - 6x^2 + 12x - 8) + B(x^2 - 4x + 4) + C(x-2) + D,$$

hence the system

$$\begin{aligned}
 1 &= A, \\
 -2 &= -6A + B, \\
 -9 &= 12A - 4B + C, \\
 16 &= -8A + 4B - 2C + D.
 \end{aligned}$$

This solves easily as

$$A = 1, B = 4, C = -5, D = -2.$$

Therefore,

$$\begin{aligned}
 \int \frac{x^3 - 2x^2 - 9x + 16}{x^3 - 6x^2 + 12x - 8} dx &= \int 1 + \frac{4}{x-2} - \frac{5}{(x-2)^2} - \frac{2}{(x-2)^3} dx \\
 &= x + 4\ln|x-2| + \frac{5}{x-2} + \frac{2}{2(x-2)^2} + C.
 \end{aligned}$$

(d): The discriminant of the numerator is $(-6)^2 - 4 \cdot 18 = -36 < 0$, so it doesn't factor further. The degree of the numerator matches the degree of the denominator, so we will decompose the integrand as

$$\frac{x^2 - 4x + 14}{x^2 - 6x + 18} = A + \frac{Bx + C}{x^2 - 6x + 18}$$

which becomes

$$x^2 - 4x + 14 = A(x^2 - 6x + 18) + Bx + C.$$

We can easily see that this solves as

$$A = 1, \quad B = 2, \quad C = -4,$$

so that

$$\int \frac{x^2 - 4x + 14}{x^2 - 6x + 18} dx = \int 1 + \frac{2x - 4}{x^2 - 6x + 18} dx$$

To handle the second term, we note that a substitution $u = x^2 - 6x + 18$ gives $du = 2x - 6 dx$, so we want to decompose the second term as

$$\frac{2x - 4}{x^2 - 6x + 18} = \frac{D(2x - 6)}{x^2 - 6x + 18} + \frac{E}{x^2 - 6x + 18}.$$

Again, this solves easily as

$$D = 1, \quad E = 2.$$

Therefore we have

$$\begin{aligned} \int \frac{x^2 - 4x + 14}{x^2 - 6x + 18} dx &= \int 1 + \frac{2x - 3}{x^2 - 6x + 18} + \frac{2}{x^2 - 6x + 18} dx \\ &= \int 1 dx + \int \frac{1}{u} dx + \int \frac{2}{9((x/3 - 1)^2 + 1)} dx \\ &= x + \ln |u| - \frac{2}{9} \cdot 3 \tan^{-1}(x/3 - 1) + C \\ &= x + \ln(x^2 - 6x + 18) + \frac{2}{3} \tan^{-1}(x/3 - 1) + C. \end{aligned}$$