3. 极坐标下二重积分的计算

若积分区域(σ)在极坐标下可由不等式

$$\rho_1(\theta)\leqslant\rho\leqslant\rho_2(\theta),\quad \alpha\leqslant\theta\leqslant\beta$$

来刻画,则二重积分可以化为如下的累次积分:

$$\iint\limits_{(\sigma)} f(x,y)\mathrm{d}\sigma=\iint\limits_{(\sigma)} f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho\mathrm{d}\theta=\int_\alpha^\beta\mathrm{d}\theta\int_{\rho_1(\theta)}^{\rho_2(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho.$$

4. 二重积分的一般换元法

考虑如下正则变换

$$u=u(x,y),\quad v=v(x,y),\quad (x,y)\in(\sigma)\subseteq\mathbf{R}^2,$$

且有$(u,v)\in(\sigma')\subseteq\mathbf{R}^2$ 成立. 其逆变换为

$$x=x(u,v),\quad y=y(u,v),\quad (u,v)\in(\sigma')\subseteq\mathbf{R}^2,$$

则有

$$\iint\limits_{(\sigma)} f(x,y)\mathrm{d}\sigma=\iint\limits_{(\sigma')} f[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}\sigma',$$

其中$\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$为变换的雅可比行列式.

8. 将下列直角坐标系下的累次积分化为极坐标下的累次积分：

(1) $\int_0^{2a}dx\int_0^{\sqrt{2ax-x^2}}f(x^2+y^2)dy,(a>0)$；　　(2) $\int_1^2 dy\int_0^y f\left(\frac{x\sqrt{x^2+y^2}}{y}\right)dx$；

(3) $\int_0^1 dy\int_{-y}^{\sqrt{y}}f(x,y)dx$；　　(4) $\int_{-a}^{a}dx\int_a^{a+\sqrt{a^2-x^2}}f(x,y)dy,(a>0)$.

解　(1) 积分区域边界曲线为 $y=\sqrt{2ax-x^2}\Leftrightarrow y^2+(x-a)^2=a^2(y\geqslant 0)$，由此可知，积分区域为圆心在$(a,0)$半径为 a 的圆.

首先，将积分区域半圆$\{0\leqslant y\leqslant\sqrt{2ax-x^2},\ 0\leqslant x\leqslant 2a\}$化为极坐标下的形式. 由 $y=\sqrt{2ax-x^2}\Leftrightarrow\rho=2a\cos\theta$，有

$$(\sigma)=\{(x,y)\mid 0\leqslant\rho\leqslant 2a\cos\theta,\ 0\leqslant\theta\leqslant\frac{\pi}{2}\},$$

则有

$$\int_0^{2a}dx\int_0^{\sqrt{2ax-x^2}}f(x^2+y^2)dy=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}f(\rho^2)\rho d\rho.$$

(2) $\int_1^2 \mathrm{d}y \int_0^y f\left(\frac{x\sqrt{x^2+y^2}}{y}\right)\mathrm{d}x$；

(2) 积分区域为$(\sigma)=\{(x,y)\mid 0\leqslant x\leqslant y,\ 1\leqslant y\leqslant 2\}$，由 $x=\rho\cos\theta$，$y=\rho\sin\theta$ 可得

$$\int_1^2 \mathrm{d}y \int_0^y f\left(\frac{x\sqrt{x^2+y^2}}{y}\right)\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{1}{\sin\theta}}^{\frac{2}{\sin\theta}} \rho f(\rho\cot\theta)\,\mathrm{d}\rho.$$

(3) $\int_0^1 \mathrm{d}y \int_{-y}^{\sqrt{y}} f(x,y)\,\mathrm{d}x$；

(3) 积分区域为$(\sigma)=\{(x,y)\mid 0\leqslant y\leqslant 1,-y\leqslant x\leqslant\sqrt{y}\}$，如 A 组题 8(3)图所示. 由 $x=\rho\cos\theta$，$y=\rho\sin\theta$ 可得直线 $x=\sqrt{y}$的方程为 $\rho=\frac{\sin\theta}{\cos^2\theta}$，$y=1$ 的方程为 $\rho=\frac{1}{\sin\theta}$.

故而

$$\int_0^1 \mathrm{d}y \int_{-y}^{\sqrt{y}} f(x,y)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{\sin\theta}{\cos^2\theta}} f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \mathrm{d}\varphi \int_0^{\frac{1}{\sin\theta}} f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho.$$

(4) $\int_{-a}^{a} \mathrm{d}x \int_a^{a+\sqrt{a^2-x^2}} f(x,y)\,\mathrm{d}y$，$(a>0)$.

积分区域$(\sigma)=\{(x,y)\mid -a\leqslant x\leqslant a,\theta\leqslant y\leqslant\theta+\sqrt{a^2-x^2}\}$

$x=\rho\cos\theta,y=\rho\sin\theta$ 可得曲线 $y=a+\sqrt{a^2-x^2}\Leftrightarrow x^2+y^2=2ay$，它的方程为 $\rho=2a\sin\theta$，$y=a$ 的方程为 $\rho=\dfrac{a}{\sin\theta}$.

故而

$$\int_{-a}^{a}\mathrm{d}x\int_{a}^{a+\sqrt{a^2-x^2}}f(x,y)\mathrm{d}x=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\mathrm{d}\theta\int_{\frac{a}{\sin\theta}}^{2a\sin\theta}f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho.$$

10. 用极坐标计算下列二重积分：

(1) $\iint\limits_{(\sigma)}(x^2+y^2)\mathrm{d}\sigma$，其中积分区域$(\sigma)=\{(x,y)\mid x^2+y^2\leqslant 9\}$；

解 (1) $\iint\limits_{(\sigma)}(x^2+y^2)\mathrm{d}\sigma=\int_0^{2\pi}\mathrm{d}\theta\int_0^3\rho^3\mathrm{d}\rho=2\pi\left(\dfrac{\rho^4}{4}\right)\Big|_0^3=2\pi\times\dfrac{3^4}{4}=\dfrac{81\pi}{2}$.

(2) $\iint\limits_{(\sigma)} e^{x^2+y^2} d\sigma$，其中积分区域$(\sigma)=\{(x,y)\mid a^2\leqslant x^2+y^2\leqslant b^2\}$，其中 $a>0,b>0$；

(2) $\iint\limits_{(\sigma)} e^{(x^2+y^2)} d\sigma = \int_0^{2\pi} d\theta \int_a^b e^{\rho^2} \rho d\rho = 2\pi\left[\frac{e^{\rho^2}}{2}\right]\Big|_a^b = \pi(e^{b^2} - e^{a^2})$.

(6) $\iint\limits_{(\sigma)} \arctan\frac{y}{x} d\sigma$，其中积分区域$(\sigma)$是圆域 $x^2+y^2\leqslant 1$ 落在第一象限的部分；

(6) $\iint\limits_{(\sigma)} \arctan\frac{y}{x} d\sigma = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \rho\theta d\rho = \int_0^{\frac{\pi}{2}} \frac{\theta}{2} d\theta = \frac{\theta^2}{4}\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{16}$.

(7) $\iint\limits_{(\sigma)} \sqrt{R^2-x^2-y^2}\mathrm{d}\sigma$,其中积分区域$(\sigma)$是圆域 $x^2+y^2\leqslant Rx$ 落在第一象限的部分;

(7) $$\iint\limits_{(\sigma)} \sqrt{R^2-x^2-y^2}\mathrm{d}\sigma=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{R\cos\theta}\sqrt{R^2-\rho^2}\rho\,\mathrm{d}\rho=-\frac{1}{3}\int_0^{\frac{\pi}{2}}(R^2-\rho^2)^{\frac{3}{2}}\Big|_0^{R\cos\theta}\mathrm{d}\theta$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{2}}R^3(1-\sin^3\theta)\mathrm{d}\theta$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{2}}R^3\mathrm{d}\theta-\frac{1}{3}R^3\int_0^{\frac{\pi}{2}}\sin^3\theta\mathrm{d}\theta$$

$$=\frac{\pi}{6}R^3-\frac{1}{3}R^3\int_0^{\frac{\pi}{2}}\sin^3\theta\mathrm{d}\theta$$

$$=\frac{\pi}{6}R^3-\frac{1}{3}R^3\int_0^{\frac{\pi}{2}}(\cos^2\theta-1)\mathrm{d}\cos\theta$$

$$=\frac{\pi}{6}R^3-\frac{1}{3}R^3\left(\cos\theta-\frac{1}{3}\cos^3\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{R^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

(8) $\iint\limits_{(\sigma)}(x+y)\mathrm{d}\sigma$,其中积分区域$(\sigma)$是圆域 $x^2+y^2\leqslant x+y$ 的内部.

$$(8)\ \iint\limits_{(\sigma)}(x+y)\mathrm{d}\sigma=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\mathrm{d}\theta\int_{0}^{\cos\theta+\sin\theta}\rho^{2}(\cos\theta+\sin\theta)\mathrm{d}\rho$$

$$=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}(\cos\theta+\sin\theta)\left(\frac{\rho^{3}}{3}\bigg|_{0}^{\cos\theta+\sin\theta}\right)\mathrm{d}\theta$$

$$=\frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}(\cos\theta+\sin\theta)^{4}\mathrm{d}\theta$$

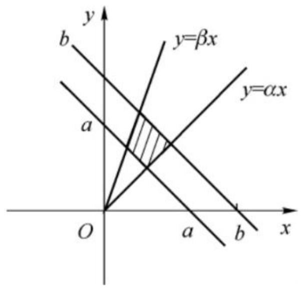
$$=\frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{3}{2}+2\sin 2\theta-\frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$$

$$=\frac{1}{3}\left(\frac{3}{2}\theta-\cos 2\theta-\frac{1}{8}\sin 4\theta\right)\bigg|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$=\frac{\pi}{2}.$$

11. 求下列各组曲线所围区域的面积：

（1）$x+y=a$，$x+y=b$，$y=\alpha x$ 和 $y=\beta x$（$0<a<b,0<\alpha<\beta$）；



解 设 $u=x+y,v=\dfrac{y}{x}$,则曲线所围区域(σ)可表示为$\{u=a,u=b,v=\alpha,v=\beta\}$. 从而

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & 1\\ -\dfrac{y}{x^2} & \dfrac{1}{x}\end{vmatrix}=\frac{x+y}{x^2},\quad \frac{\partial(x,y)}{\partial(u,v)}=\frac{x^2}{x+y}=\frac{u}{(1+v)^2},$$

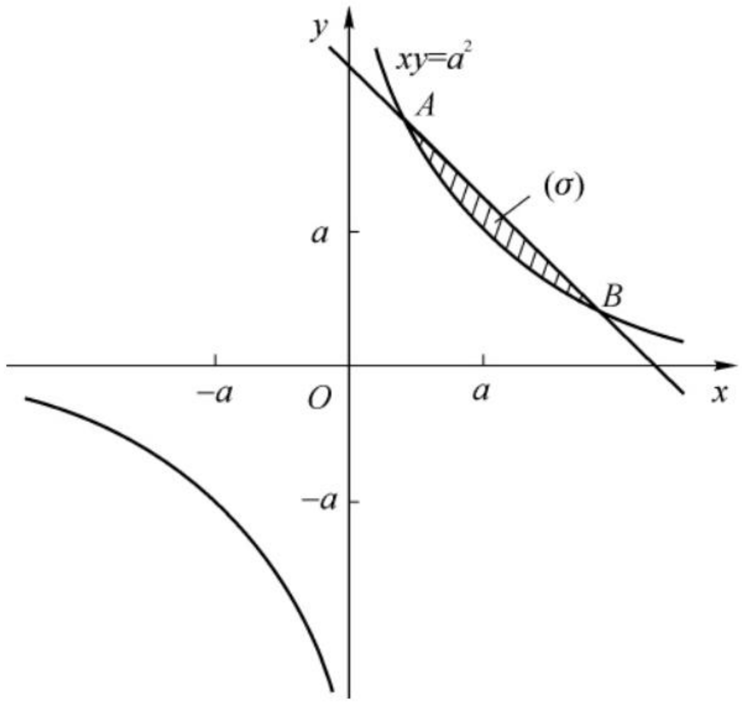
因此

$$\iint\limits_{(\sigma)}\mathrm{d}\sigma=\int_a^b\mathrm{d}u\int_\alpha^\beta\frac{u}{(1+v)^2}\mathrm{d}v=\frac{u^2}{2}\Bigg|_a^b\left(-\frac{1}{1+v}\Bigg|_\alpha^\beta\right)=\frac{1}{2}(b^2-a^2)\left(\frac{1}{1+\alpha}-\frac{1}{1+\beta}\right).$$

$$\iint_{(\sigma)} f(x,y)\mathrm{d}\sigma = \iint_{(\sigma')} f[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}\sigma',$$

其中$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$为变换的雅可比行列式.

（2）$xy=a^2$，$x+y=3a$（$a>0$）；



易知两曲线 $xy=a^2, x+y=3a$ 的交点坐标为

$$A\left(\frac{(3-\sqrt{5})a}{2},\frac{(3+\sqrt{5})a}{2}\right)\text{和 } B\left(\frac{(3+\sqrt{5})a}{2},\frac{(3-\sqrt{5})a}{2}\right)$$

$$A=\iint\limits_{(\sigma)}\mathrm{d}\sigma=\int_{\frac{(3-\sqrt{5})a}{2}}^{\frac{(3+\sqrt{5})a}{2}}\mathrm{d}x\int_{\frac{a^2}{x}}^{3a-x}\mathrm{d}y$$

$$=\int_{\frac{(3-\sqrt{5})a}{2}}^{\frac{(3+\sqrt{5})a}{2}}\left(3a-x-\frac{a^2}{x}\right)\mathrm{d}x=\left(3ax-\frac{x^2}{2}-a^2\ln x\right)\Bigg|_{\frac{(3-\sqrt{5})a}{2}}^{\frac{(3+\sqrt{5})a}{2}}$$

$$=\frac{3\sqrt{5}}{2}a^2-a^2\ln\frac{3+\sqrt{5}}{3-\sqrt{5}}.$$

12. 求由下列各族曲面所围成的立体的体积：

（1）$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,(a>0,b>0,c>0),x=0,y=0,z=0$；

V

$$=\int_0^a \mathrm{d}x\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)\mathrm{d}y$$

$$=bc\int_0^a\left[\left(1-\frac{x}{a}\right)^2-\frac{\left(1-\frac{x}{a}\right)^2}{2}\right]\mathrm{d}x=-\frac{abc}{2}\cdot\frac{\left(1-\frac{x}{a}\right)^3}{3}\Bigg|_0^a=\frac{1}{6}abc.$$

1. 三重积分的概念

(1) 三重积分的定义

设三元函数 $f(x,y,z)$是空间闭区域(V)上的有界函数,将(V)任意分成 n 个小闭区域

$$(\Delta v_1),(\Delta v_2),\cdots,(\Delta v_n),$$

其中,(Δv_i)表示第 i 个小闭区域,用 Δv_i 表示它的体积. 在每个(Δv_i)上任取一点(ξ_i,η_i,ζ_i),作乘积 $f(\xi_i,\eta_i,\zeta_i)\Delta v_i(i=1,2,\cdots,n)$,并作和 $\sum\limits_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i$. 如果当各个小闭区域直径中的最大值 λ 趋于零时,该和式的极限总存在(与子区域(Δv_i)的分法及点(ξ_i,η_i,ζ_i)的取法均无关),则称此极限值为函数 $f(x,y,z)$在闭区域(V)上的**三重积分**,记作

$$\iiint\limits_{(V)} f(x,y,z)\mathrm{d}V,$$

即

$$\iiint\limits_{(V)} f(x,y,z)\mathrm{d}V=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i,$$

其中,$f(x,y,z)$称为被积函数,$f(x,y,z)\mathrm{d}V$ 称为被积表达式,$\mathrm{d}V$ 称为体积元素,x、y 与 z 称为积分变量,(V)称为积分区域,$\sum\limits_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i$ 称为积分和.

(2) 三重积分存在定理

若三元函数 $f(x,y,z)$在空间闭区域(V)上连续，则三重积分 $\iiint\limits_{(V)} f(x,y,z)\mathrm{d}V$ 一定存在.

(3) 三重积分物理意义

设有一在 $Oxyz$ 空间上闭区域(V)上的物体，其在(V)内点(x,y,z)处的体密度为连续函数 $\rho(x,y,z)$，则该物体的质量 M 可用三重积分表示为

$$M=\iiint\limits_{(V)}\rho(x,y,z)\mathrm{d}V.$$

2. 三重积分的性质

二重积分的性质可推广到三重积分，考虑在空间有界闭区域(V)上可积的三元函数 $f(x,y,z)$与 $g(x,y,z)$，有如下性质：

(1) 线性性质

① $\iiint\limits_{(V)} kf(x,y,z)\mathrm{d}V=k\iiint\limits_{(V)} f(x,y,z)\mathrm{d}V$，$k$ 是常数；

② $\iiint\limits_{(V)}[f(x,y,z)\pm g(x,y,z)]\mathrm{d}V=\iiint\limits_{(V)} f(x,y,z)\mathrm{d}V\pm\iiint\limits_{(V)} g(x,y,z)\mathrm{d}V.$

(2) 积分区域可加性

$$\iiint\limits_{(V)} f(x,y,z)\,\mathrm{d}V = \iiint\limits_{(V_1)} f(x,y,z)\,\mathrm{d}V + \iiint\limits_{(V_2)} f(x,y,z)\,\mathrm{d}V.$$

（3）积分不等式

① 若 $f(x,y,z) \leqslant g(x,y,z)$，$\forall (x,y,z) \in (V)$，则

$$\iiint\limits_{(V)} f(x,y,z)\,\mathrm{d}V \leqslant \iiint\limits_{(V)} g(x,y,z)\,\mathrm{d}V;$$

② $\left|\iiint\limits_{(V)} f(x,y,z)\,\mathrm{d}V\right| \leqslant \iiint\limits_{(V)} |f(x,y,z)|\,\mathrm{d}V;$

③ 若 $l \leqslant f(x,y,z) \leqslant L$，$\forall (x,y,z) \in (V)$，则

$$lV \leqslant \iiint\limits_{(V)} f(x,y,z)\,\mathrm{d}V \leqslant LV,$$

1．计算下列三重积分：

(1) $\iiint\limits_{(V)} xyz\mathrm{d}V$，其中积分区域$(V)=\{(x,y,z)\mid 0\leqslant x\leqslant 1,0\leqslant y\leqslant 1,0\leqslant z\leqslant 1\}$；

(2) $\iiint\limits_{(V)} (x+y+z)\mathrm{d}V$，其中积分区域$(V)=\{(x,y,z)\mid 0\leqslant x\leqslant 1,0\leqslant y\leqslant 2,0\leqslant z\leqslant 3\}$；

解 (1) 由积分区域$(V)=\{(x,y,z)\mid 0\leqslant x\leqslant 1,0\leqslant y\leqslant 1,0\leqslant z\leqslant 1\}$，可得

$$\iiint\limits_{(V)}(xyz)\mathrm{d}V=\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1(xyz)\mathrm{d}z=\int_0^1 x\mathrm{d}x\int_0^1 y\mathrm{d}y\int_0^1 z\mathrm{d}z$$

$$=\left(\frac{x^2}{2}\right)\Bigg|_0^1\cdot\left(\frac{y^2}{2}\right)\Bigg|_0^1\cdot\left(\frac{z^2}{2}\right)\Bigg|_0^1=\frac{1}{8}.$$

（2）由积分区域$(V)=\{(x,y,z)\mid 0\leqslant x\leqslant 1,0\leqslant y\leqslant 2,0\leqslant z\leqslant 3\}$，可得

$$\begin{aligned}\iiint\limits_{(V)}(x+y+z)\mathrm{d}V&=\int_0^1\mathrm{d}x\int_0^2\mathrm{d}y\int_0^3(x+y+z)\mathrm{d}z\\&=\int_0^1\mathrm{d}x\int_0^2\left(xz+yz+\frac{1}{2}z^2\right)\Big|_0^3\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^2\left(3x+3y+\frac{9}{2}\right)\mathrm{d}y\\&=\int_0^1\left(3xy+\frac{3y^2}{2}+\frac{9y}{2}\right)\Big|_0^2\mathrm{d}x=\int_0^1(6x+5)\mathrm{d}x\\&=(3x^2+15x)\Big|_0^1=18.\end{aligned}$$

(3) $\iiint\limits_{(V)} y\mathrm{d}V$,其中积分区域$(V)$是由抛物柱面 $y=\sqrt{x}$,平面 $y=0$,$z=0$ 与 $x+z=\frac{\pi}{2}$ 所围成的区域;

(3) 由积分区域$(V)=\left\{(x,y,z)\mid 0\leqslant x\leqslant\frac{\pi}{2},0\leqslant y\leqslant\sqrt{x},0\leqslant z\leqslant\frac{\pi}{2}-x\right\}$,可得

$$\iiint\limits_{(V)} y\mathrm{d}V=\int_0^{\frac{\pi}{2}}\mathrm{d}x\int_0^{\sqrt{x}}\mathrm{d}y\int_0^{\frac{\pi}{2}-x}y\mathrm{d}z$$

$$=\int_0^{\frac{\pi}{2}}\mathrm{d}x\int_0^{\sqrt{x}}y\left(\frac{\pi}{2}-x\right)\mathrm{d}y=\int_0^{\frac{\pi}{2}}\left[\frac{y^2}{2}\left(\frac{\pi}{2}-x\right)\right]\Big|_0^{\sqrt{x}}\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\pi x}{4}-\frac{x^2}{2}\right)\mathrm{d}x=\left(\frac{\pi x^2}{8}-\frac{x^3}{6}\right)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{8}\times\frac{\pi^2}{4}-\frac{1}{6}\frac{\pi^3}{8}=\frac{\pi^3}{96}.$$

(4) $\iiint\limits_{(V)} z\mathrm{d}V, (V)=\left\{(x, y, z) \mid x^2+y^2<2x, 0\leqslant z\leqslant \sqrt{4-x^2-y^2}\right\}$;

（方法二）在柱坐标下计算，可得

$$(V)=\{(\rho,\theta,z)\mid 0\leqslant\theta\leqslant 2\pi, \rho\leqslant 2\cos\theta, 0\leqslant z\leqslant\sqrt{4-\rho^2}\},$$

所以

$$\iiint\limits_{(V)} z\mathrm{d}V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}\mathrm{d}\rho\int_0^{\sqrt{4-\rho^2}} z\rho\,\mathrm{d}z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}\left(\frac{z^2}{2}\rho\right)\Big|_0^{\sqrt{4-\rho^2}}\mathrm{d}\rho=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}\rho\,\frac{4-\rho^2}{2}\mathrm{d}\rho$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\frac{(4-4\cos^2\theta)^2-16}{8}\mathrm{d}\theta=2\int_0^{\frac{\pi}{2}}\left[2-\frac{16(1-\cos^2\theta)^2}{8}\right]\mathrm{d}\theta=2\pi-4\int_0^{\frac{\pi}{2}}\sin^4\theta\mathrm{d}\theta$$

$$=2\pi-\int_0^{\frac{\pi}{2}}(1+2\cos 2\theta+\frac{1+\cos 4\theta)^2}{2})\mathrm{d}\theta$$

$$=2\pi-(\frac{3}{2}\theta+\sin 2\theta+\sin 4\theta)\Big|_0^{\frac{\pi}{2}}$$

$$=2\pi-4\times\frac{3}{4}\times\frac{1}{2}\frac{\pi}{2}=\frac{5}{4}\pi.$$

（5）$\iiint\limits_{(V)} z^2\mathrm{d}V$，其中积分区域$(V)$是由 $x^2+y^2+z^2=4$ 与 $x^2+y^2+z^2=4z$ 所围成的区域；

（5）由于积分区域(V)是由 $x^2+y^2+z^2=4$ 与 $x^2+y^2+z^2=4z$ 所围成，所以可得

$$\begin{aligned}(V)&=\{(x,y,z)\mid 0<z<2,(x,y)\in(\sigma_z)\}\\&=\{(x,y,z)\mid 0<z<1,(x,y)\in(\sigma_{z1})\}\cup\{(x,y,z)\mid 1<z<2,(x,y)\in(\sigma_{z2})\},\end{aligned}$$

其中$(\sigma_{z1})=\{(x,y)\mid x^2+y^2\leqslant 4-(z-2)^2\}$，$(\sigma_{z2})=\{(x,y)\mid x^2+y^2\leqslant 4-z^2\}$，由此可得

$$I=\iiint\limits_{(V)} z^2\,\mathrm{d}V=\int_0^1 z^2\Big(\iint\limits_{(\sigma_{z1})}\mathrm{d}\sigma\Big)\mathrm{d}z+\int_1^2 z^2\Big(\iint\limits_{(\sigma_{z2})}\mathrm{d}\sigma\Big)\mathrm{d}z.$$

由$\iint\limits_{(\sigma_{z1})}\mathrm{d}\sigma$和$\iint\limits_{(\sigma_{z2})}\mathrm{d}\sigma$的几何意义为平面区域$(\sigma_{z1})$和$(\sigma_{z2})$的面积，易知

$$\iint\limits_{(\sigma_{z1})}\mathrm{d}\sigma=\pi[4-(z-2)^2],\quad \iint\limits_{(\sigma_{z2})}\mathrm{d}\sigma=\pi(4-z^2),$$

所以

$$\begin{aligned}I&=\pi\int_0^1 z^2[4-(z-2)^2]\mathrm{d}z+\pi\int_1^2 z^2(4-z^2)\mathrm{d}z\\&=\pi\Big(z^4-\frac{1}{5}z^5\Big)\Big|_0^1+\pi\Big(\frac{4}{3}z^3-\frac{1}{5}z^5\Big)\Big|_1^2\\&=\frac{4\pi}{5}+\frac{47\pi}{15}=\frac{59\pi}{15}.\end{aligned}$$

(6) $\iiint\limits_{(V)} x^2 \mathrm{d}V$,其中积分区域$(V)$是由 $x^2+y^2+z^2=1$ 与 $x^2+y^2+z^2=2x$ 所围成的区域;

（6）由积分区域(V)是由 $x^2+y^2+z^2=1$ 与 $x^2+y^2+z^2=2x$ 所围成的区域，可得

$$(\sigma_{x1})=\{(y,z)\mid y^2+z^2\leqslant 1-(x-1)^2\},\quad (\sigma_{x2})=\{(y,z)\mid y^2+z^2\leqslant 1-x^2\},$$

由此可知：

$$I=\iiint\limits_{(V)}x^2\,\mathrm{d}V=\int_0^{\frac{1}{2}}x^2\Big(\iint\limits_{(\sigma_{x_1})}\mathrm{d}\sigma\Big)\mathrm{d}x+\int_{\frac{1}{2}}^{1}x^2\Big(\iint\limits_{(\sigma_{x_2})}\mathrm{d}\sigma\Big)\mathrm{d}x.$$

由 $\iint\limits_{(\sigma_{x_1})}\mathrm{d}\sigma$ 和 $\iint\limits_{(\sigma_{x_2})}\mathrm{d}\sigma$ 的几何意义为平面区域(σ_{x_1})和(σ_{x_2})的面积，易知

$$\iint\limits_{(\sigma_{x_1})}\mathrm{d}\sigma=\pi[1-(x-1)^2],\quad \iint\limits_{(\sigma_{x_2})}\mathrm{d}\sigma=\pi(1-x^2),$$

所以

$$\begin{aligned}I&=\pi\int_0^{\frac{1}{2}}x^2[1-(x-1)^2]\mathrm{d}x+\pi\int_{\frac{1}{2}}^{1}x^2(1-x^2)\mathrm{d}x\\&=\pi\int_0^{\frac{1}{2}}(2x^3-x^4)\mathrm{d}x+\pi\int_{\frac{1}{2}}^{1}(x^2-x^4)\mathrm{d}x\\&=\pi\left(\frac{1}{2}x^4-\frac{1}{5}x^5\right)\Big|_0^{\frac{1}{2}}+\pi\left(\frac{1}{3}x^3-\frac{1}{5}x^5\right)\Big|_{\frac{1}{2}}^{1}\\&=\frac{\pi}{40}+\frac{47\pi}{480}=\frac{59\pi}{480}.\end{aligned}$$

(7) $\iiint\limits_{(V)} xyz\mathrm{dV}$,其中积分区域$(V)$是由 $x^2+y^2+z^2=1$, $x=0$, $y=0$ 及 $z=0$ 所围成的且在第一卦限内的区域;

(7)(方法一) 由积分区域(V)是由 $x^2+y^2+z^2=1$, $x=0$, $y=0$ 及 $z=0$ 所围成的第一卦限内的区域,可知(V)为八分之一的球体.

令

$$(D)=\{(x,y)\mid x^2+y^2\leqslant 1, x\geqslant 0, y\geqslant 0\}=\{(x,y)\mid 0\leqslant x\leqslant 1, -0\leqslant y\leqslant \sqrt{1-x^2}\},$$

用直角坐标计算可得

$$
\begin{aligned}
\iiint\limits_{(V)} xyz\,\mathrm{d}V &= \iint\limits_{(D)} xy\,\mathrm{d}x\mathrm{d}y\int_0^{\sqrt{1-x^2-y^2}} z\mathrm{d}z = \frac{1}{2}\iint\limits_{(D)} xy(1-x^2-y^2)\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 x\mathrm{d}x\int_0^{\sqrt{1-x^2}}[(1-x^2)y-y^3]\mathrm{d}y \\
&= \frac{1}{2}\int_0^1\left[\frac{1}{2}x(1-x^2)-\frac{1}{2}x^3(1-x^2)-\frac{1}{4}x(1-x^2)^2\right]\mathrm{d}x \\
&= \frac{1}{48}.
\end{aligned}
$$

（方法二） 在球坐标下计算，积分区域可以表示为

$$(V)=\{(r,\theta,\varphi)\mid 0\leqslant\theta\leqslant\frac{\pi}{2},0\leqslant\varphi\leqslant\frac{\pi}{2},0\leqslant r\leqslant 1\},$$

由

$$\begin{cases}x=r\cos\theta\sin\varphi,\\ y=r\sin\theta\sin\varphi,\ \mathrm{d}V=r^2\sin\varphi\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r\\ z=r\cos\varphi,\end{cases}$$

可得

$$\begin{aligned}\iiint\limits_{(V)}xyz\,\mathrm{d}V&=\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 r^5\cos\varphi\sin\theta\cos\theta\sin^3\varphi\mathrm{d}r\\&=\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\mathrm{d}\theta\int_0^1 r^5\,\mathrm{d}r\\&=\frac{1}{6}\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\mathrm{d}\theta\end{aligned}$$

$$= \frac{1}{6} \times \frac{1}{2} \int_{0}^{\frac{\pi}{2}} \cos \varphi \sin^{3} \varphi \mathrm{d}\varphi$$

$$= \frac{1}{6} \times \frac{1}{2} \int_{0}^{\frac{\pi}{2}} \sin^{3} \varphi \mathrm{d} \sin \varphi$$

$$= \frac{1}{6} \times \frac{1}{2} \times \frac{1}{4} (\sin^{4} \varphi) \Big|_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{6} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{48}.$$

(8) $\iiint\limits_{(V)} xy\,\mathrm{d}V$，其中积分区域$(V)$是由 $x^2+y^2=1$ 和平面 $z=0,z=1,x=0,y=0$ 所围成的且在第一卦限内的区域.

(8) 由于(V)是由柱面 $x^2+y^2=1$ 和平面 $z=0,z=1,x=0,y=0$ 所围成的第一卦限内的区域，在直角坐标下和柱面坐标下，该区域分别可以表示为

$$(V)=\{(x,y,z)\mid 0\leqslant z\leqslant 1,0\leqslant x\leqslant 1,0\leqslant y\leqslant \sqrt{1-x^2}\}$$

$$=\{(\rho,\theta,z)\mid 0\leqslant\theta\leqslant\frac{\pi}{2},0\leqslant\rho\leqslant 1,0\leqslant z\leqslant 1\},$$

故而，在直角坐标系下计算有

$$\iiint\limits_{(V)} xy\,\mathrm{d}V = \int_0^1 x\mathrm{d}x\int_0^{\sqrt{1-x^2}} y\mathrm{d}y\int_0^1 \mathrm{d}z = \int_0^1 x\mathrm{d}x\int_0^{\sqrt{1-x^2}} y\mathrm{d}y$$

$$= \int_0^1 x\,\frac{(\sqrt{1-x^2})^2}{2}\mathrm{d}x = \frac{1}{2}\int_0^1 (x-x^3)\mathrm{d}x$$

$$= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{8}.$$

在柱面坐标系下计算有

$$\iiint\limits_{(V)} xy\,\mathrm{d}V = \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\mathrm{d}\theta\int_0^1 \rho^3\,\mathrm{d}\rho\int_0^1 \mathrm{d}z = \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\mathrm{d}\theta\int_0^1 \rho^3\,\mathrm{d}\rho$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\mathrm{d}(\sin\theta)\int_0^1 \mathrm{d}\left(\frac{\rho^4}{4}\right) = \frac{\sin^2\theta}{2}\Bigg|_0^{\frac{\pi}{2}} \cdot \frac{\rho^4}{4}\Bigg|_0^1$$

$$= \frac{1}{2}\times\frac{1}{4} = \frac{1}{8}.$$