Propagation of Uncertainties
误差传递

吴先球
华南师范大学
物理学科基础课实验教学示范中心

Propagation of Uncertainties 误差传递

- Two distinct steps for the measure of most physical quantities:
  - Directly measure one or more quantities x, y
  - Calculate the quantity $q=f(x,y)$
- The estimation of uncertainties also involves two steps:
  - Measure σx, σy
  - Calculate σq

3.1 Uncertainties in Direct Measurements直接测量中的误差

- Uncertainties in reading scale
  - Analog display
  - Digital display
- Other sources of uncertainty
  - Problem of definition
    - Locating the center of a lens
    - The image center

Repeatable Measurement

- If a measurement can repeated, it should be repeated, both to obtain a more reliable answer(by average) and , even more important, to get some estimate of the uncertainties

Counting events that occur at random

- In a sample of radioactive material, each individual nucleus decays at a random time, but there is a definite average rate at which we could expect to see decays occur in the whole sample.
- If we count the number of events in some time T and get the answer v,

(average number of events in time T)= $\boxed{v \pm \sqrt{v}}$

3.2 Sums and Differences; Products and Quotients
和差积商

Uncertainty in Sums and Differences

- If several quantities x, …, w are measured with uncertainties δx, …, δ w,
- And the measured values used to compute

  $Q=x+\ldots+z-(u+\ldots+w)$;
- Then the uncertainty in the computed value of q is the sum

  $\delta Q= \delta x+\ldots+ \delta z+(\delta u+\ldots+ \delta w)$;

Uncertainties in Sums and Difference

$$q = x + \ldots + z - \underline{(u + \ldots + w)};$$

$$\delta q = \delta x + \ldots + \delta z + \underline{\delta u + \ldots + \delta w}$$

Notice Some much smaller uncertainties make negligible contribution to the final uncertainty. Those uncertainties are negligible and can be ignored from the outset.

Products and Quotients

$$q = \frac{x * \ldots * z}{u * \ldots * w};$$

$$\frac{\delta q}{q} = \frac{\delta x}{x} + \ldots + \frac{\delta z}{z} + \frac{\delta u}{u} + \ldots + \frac{\delta w}{w}$$

Measured Quantity Times Exact Number
测量值和常量的乘积

$$q = Bx;$$

$$\delta q = |B|\delta x$$

Power 乘方

$$q = x^n;$$

$$\frac{\delta q}{q} = n\frac{\delta x}{x}$$

Unnecessarily Large Uncertainties

- Summarized rules
  - +, - :the uncertainties add
  - * , / :the fractional add

3.3 Independent Uncertainties in a Sum 求和中的独立误差

- If x and y are measured independently and our errors are random in nature, then there is 50% percent chance that an underestimate of x will be accompanied by and overestimate of y: or vice versa.

$$\delta q = \delta x + \delta y$$

Add in quadrature:

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

Independent measurement, Normal distribution 独立测量，正态分布

- If the measurements of x and y are made independently, and are both governed by the normal distribution, the the uncertainty in q=x+y is given by:

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$\sqrt{(\delta x)^2 + (\delta y)^2} \le \delta x + \delta y$$

3.4 More about Independent Uncertainties

Uncertainties in Sums and Difference

If the uncertainties in x, …,w are known to be **independent** and **random**, then the uncertainties in q is the quadratic sum of the original uncertainties:

$$q = x + \ldots + z - (u + \ldots + w);$$

$$\delta q = \sqrt{(\delta x)^2 + \ldots + (\delta z)^2 + (\delta u)^2 + \ldots + (\delta w)^2}$$

Products and Quotients

$$q = \frac{x * ... * z}{u * ... * w};$$

If the uncertainties in x, …,w are known to be independent and random, then the uncertainties in q is the quadratic sum of the original fractional uncertainties:

$$\frac{\delta q}{|q|} = \sqrt{(\frac{\delta x}{x})^2 + ... + (\frac{\delta z}{z})^2 + (\frac{\delta u}{u})^2 + ... + (\frac{\delta w}{w})^2}$$

3.5 Arbitrary Functions of One Variable
单变量时的任意函数

Uncertainties in Any Function of One Variable

- If x is measured with uncertainty δx and is used to calculate the function q(x), then the uncertainty δq is

$$\boxed{\delta q = \left|\frac{dq}{dx}\right| \delta x}$$

3.6 Propagation Step by Step

Riview

Propagation of uncertainties

If	In any case	X,y is independent and random
$q=x+y$ $(q=x-y)$	$\delta q \le \delta x + \delta y$	$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$
$q=x\times y$ $(q=x\div y)$	$\frac{\delta q}{q} \le \frac{\delta x}{x} + \frac{\delta y}{y}$	$\frac{\delta q}{q} = \sqrt{(\frac{\delta x}{x})^2 + (\frac{\delta y}{y})^2}$
$q=f(x)$ $(eg: \sin x, \sqrt{x}, x^n \ldots)$	$\delta q = \left\lvert \frac{dq}{dx} \right\rvert \delta x$	

3.7 example (i)

Measurement of g with a Simple Pendulum

$$g = 4\pi^2 l / T^2$$

$$g_{best} = f(l_{best}, T_{best}) = 4\pi^2 \times 92.95 / 1.936^2 = 979\ \mathrm{cm/sec^2}$$

$$\frac{\delta l}{l} = 0.1\% \qquad \text{and} \qquad \frac{\delta T}{T} = 0.2\%$$

$$\frac{\delta g}{g} = \sqrt{\left(\frac{\delta l}{l}\right)^2 + \left(2\frac{\delta T}{T}\right)^2}$$

3.7 example (i) $g = 4\pi^2 l / T^2$

Measurement of g with a Simple Pendulum

g is the product or quotient of three factors, $4\pi^2$ **,** l **,and** T^2

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g is the product or quotient of three factors, $4\pi^2$, l, and T^2

where $4\pi^2$ has no uncerainty

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Measurement of g with a Simple Pendulum

g is the product or quotient of three factors, $4\pi^2$, l, and T^2

where $4\pi^2$ has no uncerainty

$$\frac{\delta g}{g} = \sqrt{\left(\frac{\delta(4\pi^2)}{4\pi^2}\right)^2 + \left(\frac{\delta l}{l}\right)^2 + \left(2\frac{\delta T}{T}\right)^2} = \sqrt{\left(\frac{\delta l}{l}\right)^2 + \left(2\frac{\delta T}{T}\right)^2}$$

3.7 example (ii)

refractive Index using Snell's Law

$$n = \sin i / \cos r$$

n is the quotient of two factors, sin i and cos r

$$\delta \sin\theta = \left|\frac{\delta \sin\theta}{d\theta}\right| \delta\theta = \left|\cos\theta\right| \delta\theta$$

$$\frac{\delta n}{n} = \sqrt{\left(\frac{\delta \sin i}{\sin i}\right)^2 + \left(\frac{\delta \cos r}{\cos r}\right)^2}$$

3.7 example (ii)

refractive Index using Snell's Law

$n = \sin i / \cos r$

I(deg) all$\pm$1	R(deg) all$\pm$1	sin i	sin r	n	$\left\lvert\frac{\delta \sin i}{di}\right\rvert$	$\left\lvert\frac{\delta \sin r}{dr}\right\rvert$	$\left\lvert\frac{\delta n}{n}\right\rvert$
20	13	.342	.225	1.52	5%	8%	9%
40	23.5	.643	.399	1.61	2%	4%	5%

3.8 a more complicated example

acceleration of a cart down a slope

$$a=(\upsilon_2^2-\upsilon_1^2)/2s=\frac{1}{2s}\left(\frac{l^2}{t_2^2}-\frac{l^2}{t_1^2}\right)=\left(\frac{l^2}{2s}\right)\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)$$

$$\left.\begin{array}{l}\left.\begin{array}{l}1/\dfrac{1}{t_2^2}\\[2ex] 1/\dfrac{1}{t_1^2}\end{array}\right\}\Rightarrow\dfrac{1}{t_2^2}-\dfrac{1}{t_1^2}\ (9\%)\\[6ex] \qquad\qquad\qquad\dfrac{l^2}{2s}\ (2\%)\end{array}\right\}\Rightarrow\left(\frac{l^2}{2s}\right)\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)=\sqrt{9^2+2^2}\ \%=9\%$$

Interesting features in the example

$$\frac{1}{t_2^2}-\frac{1}{t_1^2}\ (9\%)$$

$$\left(\frac{l^2}{2s}\right)\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)=\sqrt{9^2+2^2}\ \%=9\%$$

$\Rightarrow$ **The uncertainties in *l* and *s* can be ignored**

Why?

$$t_1(2\%) \quad t_2(3\%)$$

$$\frac{1}{t_2^2}-\frac{1}{t_1^2}\ (9\%)$$

$\Rightarrow$ **The percent uncerainties in** t_1 **and** t_2 **grow when evaluate** $\frac{1}{t_2^2}-\frac{1}{t_1^2}$

How ?

Power 乘方

$$q = x^n;$$

$$\frac{\delta q}{q} = \frac{\left|\frac{\delta q}{dx}\right|\delta x}{x^n} = n\frac{\delta x}{x}$$

$$eg:\ q = T^2; \frac{\delta q}{q} = 2\frac{\delta T}{T}$$

Constant B in *absolut uncertainty and* in *fraction's*

q=Bx : $\delta q=\delta x+\delta x+...=B\delta x$

? q=Bx/y : $\boxed{\dfrac{\delta q}{q}=B\sqrt{\left(\dfrac{\delta x}{x}\right)^2+\left(\dfrac{\delta y}{y}\right)^2}}$

$$\because \delta q=B\,\delta(x/y)$$

$$\therefore \frac{\delta q}{q}=\frac{B\,\delta(x/y)}{Bx/y}=\frac{\delta(x/y)}{x/y}$$

The END